

INTERACTION OF MODULATED GRAVITY WATER WAVES OF FINITE DEPTH

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ABSTRACT. We consider the capillary-gravity water-waves problem of finite depth with a flat bottom of one or two horizontal dimensions. We derive the modulation equations of leading and next-to-leading order in the hyperbolic scaling for three weakly amplitude-modulated plane-wave solutions of the linearized problem in the absence of quadratic and cubic resonances. We fully justify the derived system of macroscopic equations in the case of pure gravity waves, i.e. in the case of zero surface tension, employing the stability of the water-waves problem on the time-scale $O(1/\epsilon)$ obtained by Alvarez-Samaniego and Lannes.

1. INTRODUCTION

A significant part in the research on water waves is based on the study of asymptotic limits derived from the original water-waves problem, which is considered as providing the complete description of their behavior. The benefit of this method is that these reduced models have, on the one hand, a clearer structure and, on the other hand, they highlight a particular qualitative feature of the wave evolution. Depending on the aspect of the nature of the water waves one is interested in, one has to employ the relevant asymptotic scaling, obtaining in each case a different macroscopic limit. Considering that from the outset the original water-waves problem has some fundamental characteristics, the result is a plethora of different equations which are presumed to describe approximatively the behavior of water waves in different situations and regarding different aspects. While the choice of the relevant asymptotic scaling requires a thorough understanding of the initial model, from an analytical point of view the crucial question is the justification of the derived model, i.e. the proof that its solutions indeed are approximations of solutions to the original problem.

Concerning the initial set-up of the water waves problem one could distinguish roughly between (a) gravity or capillary-gravity waves (the latter ones taking into account together with gravity also the surface tension as driving forces for the evolution of the waves), (b) finite- or infinite-depth water, and (c) two- or three-dimensional space, where in the former case one considers a vertical plane in the water domain that contains the (dominant) direction of evolution, assuming that the waves are (nearly) constant in the direction normal to the plane. Of course, this is only a very rough classification (e.g. in the case of finite depth one can consider shallow or deep water, flat bottoms or bottoms with some (smooth or rough) topography, or even moving bottoms etc.), but it seems to be the prevalent one in the mathematical-analytical literature, where the fundamental question concerns

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the well-posedness of the water-waves problem and in particular the existence time of its solutions.

In the case of gravity water waves in two dimensions and for infinite depth, first local well-posedness results were obtained by Nalimov [41] in 1974 for small Sobolev initial data, and by Shinbrot [48] and Kano and Nishida [32] for analytic initial data. The method of Nalimov was employed to prove local well-posedness in the case of finite depth by Yosihara [56] and by Craig [11], who obtained also first rigorous justification results of the Korteweg-de Vries (KdV) and Boussinesq approximations. However, in the case of infinite depth, the crucial breakthrough was made by the work of S. Wu, who presented local well-posedness results for the two- and three-dimensional cases without smallness assumptions on the initial data in [52], [53], which she extended to almost global and global existence, respectively, in [54], [55]. Independently, global existence in three dimensions was shown by Germain, Masmoudi, and Shatah in [20].

In the case of finite depth the first general well-posedness result for gravity waves in three dimensions was obtained by Lannes [36] in 2005. This result was extended in [2], where the characteristic dimensionless parameters of the original water-waves problem have been worked out in order to provide a stable fundament for the derivation and justification of various asymptotic limits. The results obtained, are presented in more detail in the survey [37], to which we take explicit reference in the present paper. Concerning the well-posedness of capillary-gravity water-waves, we mention exemplarily only the more recent selection [5, 45, 24, 8, 47, 10, 4, 40, 1, 21, 38], to which we refer for more details on the various results obtained, their development and their extensions.

As mentioned above, for each original water-wave problem with its own characteristics, different asymptotic limits can be obtained. The main distinction of the derived models is with respect to the shallowness parameter $\mu = H_0^2/L^2$ of the original equations, where $H_0 > 0$ is the water depth and $L = 1$ is the characteristic horizontal length-scale. For $\mu \ll 1$ we speak of shallow water, while for $\mu \approx 1$ and $\mu \geq 1$ of deep water (with the limiting case of infinite depth as $\mu \rightarrow \infty$). This classification is not arbitrary. Indeed, the main difference in the behavior of water-waves in these two cases is that for increasing water depth the rôle of dispersion becomes more dominant, see, e.g., [37, §1.3]. Of course within each of these two main classes of models, finer distinctions can be, and indeed are, made. Since in the present article we consider the deep (though finite) water case $\mu \geq 1$, we refrain to mention any of the various shallow water models, but refer to the survey [37], which seems to give a complete account of the "state-of-the-art" in 2013 concerning their derivation and justification. However, we would like to mention the justification of the celebrated KdV-equation in the two-dimensional case in [44, 45] after some first results in [11, 33], and with improvements in [7, 25, 26].

We will address modulation equations further below, after presenting in the following the capillary-gravity water waves equations.

The capillary-gravity water waves problem of finite depth $0 < \sqrt{\mu} < \infty$ with a flat bottom extending over all of \mathbb{R}^d , $d = 1, 2$, can be written in the following non-dimensionalized form, due to Zakharov [57], Craig, C. Sulem, P.-L. Sulem [14, 15, 49], and Alvarez-Samaniego, Lannes [2, 37]:

$$(1.1) \quad \partial_t U + \mathcal{N}_{\epsilon, \sigma}(U) = 0, \quad U = (\zeta, \psi)^T, \quad \mathcal{N}_{\epsilon, \sigma} = (\mathcal{N}_{\epsilon, \sigma}^1, \mathcal{N}_{\epsilon, \sigma}^2)^T,$$

where

$$(1.2) \quad \mathcal{N}_{\epsilon, \sigma}^1(U) = -\mathcal{G}[\epsilon\zeta]\psi,$$

$$(1.3) \quad \mathcal{N}_{\epsilon, \sigma}^2(U) = \zeta - \frac{1}{\text{Bo}} \nabla \cdot \left(\frac{\nabla \zeta}{\sqrt{1 + \epsilon^2 |\nabla \zeta|^2}} \right) + \frac{\epsilon}{2} |\nabla \psi|^2 - \frac{\epsilon}{2} \frac{(\mathcal{G}[\epsilon\zeta]\psi + \epsilon \nabla \zeta \cdot \nabla \psi)^2}{1 + \epsilon^2 |\nabla \zeta|^2}.$$

Here, the unknown functions of time $t \in [0, T)$, $T > 0$, and space $X \in \mathbb{R}^d$ are the surface elevation $\zeta : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ and the trace $\psi : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the velocity potential of the fluid at the surface. The scaling parameter $0 < \epsilon \leq 1$ is the steepness of the wave, i.e., the ratio of the amplitude of the surface elevation above the still water level $\zeta = 0$ to the characteristic horizontal length $L = 1$.

The second term in (1.3) corresponds to the surface tension, which is essentially the mean curvature of the surface scaled by the (inverse) Bond number $\frac{1}{\text{Bo}} = \frac{\sigma}{\rho g}$, where σ, ρ, g are the (dimensionless) coefficients of the surface tension, the fluid density and the gravity acceleration, respectively. When $\sigma = 0$, this term is absent and we speak of gravity water waves.

The most important term in the above formulation is the Dirichlet-Neumann operator

$$(1.4) \quad \mathcal{G}[\epsilon\zeta]\psi = \partial_z \Phi(\cdot, \epsilon\zeta) - \nabla(\epsilon\zeta) \cdot \nabla \Phi(\cdot, \epsilon\zeta) = \sqrt{1 + |\nabla(\epsilon\zeta)|^2} \partial_{\mathbf{n}} \Phi(\cdot, \epsilon\zeta)$$

where the velocity potential of the fluid Φ solves the boundary value problem for the Laplace equation

$$(1.5) \quad \begin{cases} \Delta_{X,z} \Phi = 0, & -\sqrt{\mu} \leq z \leq \epsilon\zeta, \\ \Phi(\cdot, \epsilon\zeta) = \psi, & \partial_z \Phi(\cdot, -\sqrt{\mu}) = 0 \end{cases}$$

in the fluid domain at time $t \geq 0$,

$$\Omega_{\epsilon, t} = \{(X, z) \in \mathbb{R}^{d+1} : -\sqrt{\mu} \leq z \leq \epsilon\zeta(t, X)\},$$

with Dirichlet data ψ at the surface and Neumann boundary data at the bottom. With \mathbf{n} in (1.4) being the upward unit normal vector at the surface $\epsilon\zeta$, we see that the Dirichlet-Neumann operator $\mathcal{G}[\epsilon\zeta]$ relates the Dirichlet data ψ to the normal derivative of the potential Φ at the surface, thus justifying its name. In particular, the first equation of the system (1.1),

$$(1.6) \quad \partial_t \zeta - \mathcal{G}[\epsilon\zeta]\psi = 0,$$

codifies the physical assumption that fluid particles at the surface stay there for all times. We note also that the Dirichlet-Neumann operator is linear in ψ but nonlinear in ζ .

The second equation of the gravity water-waves problem (1.1) (with $\sigma = 0$),

$$(1.7) \quad \partial_t \psi + \zeta + \frac{\epsilon}{2} |\nabla \psi|^2 - \frac{\epsilon}{2} \frac{(\mathcal{G}[\epsilon\zeta]\psi + \epsilon \nabla \zeta \cdot \nabla \psi)^2}{1 + \epsilon^2 |\nabla \zeta|^2} = 0,$$

originates from the Euler equation for the fluid velocity $\nabla_{X,z}(\epsilon\Phi)$ of an inviscid, homogeneous, incompressible, and irrotational fluid in $\Omega_{\epsilon, t}$ under the influence of gravity and with constant external (atmospheric) pressure at the surface. Integrating the Euler equation over the space variables (X, z) and evaluating it at the surface $z = \epsilon\zeta$, one gets

$$(1.8) \quad \partial_t(\epsilon\Phi)(\cdot, \epsilon\zeta) + \frac{1}{2} |\nabla_{X,z}(\epsilon\Phi)(\cdot, \epsilon\zeta)|^2 + \epsilon\zeta = 0.$$

By use of the chain rule on $\psi = \Phi(\cdot, \epsilon\zeta)$, one obtains with (1.4) and (1.6) that (1.8) is equivalent to (1.7). Thus, determining $U = (\zeta, \psi)$ via the system (1.6), (1.7), we have all required data to solve (1.5) for Φ (under reasonable regularity assumptions on U and under the condition that the flow is at rest as $|(X, z)| \rightarrow \infty$). From the Euler equation we can then determine also the pressure of the fluid. It was Zakharov who noted in [57] that the knowledge of $U = (\zeta, \psi)$ is sufficient for solving the water-waves problem in this way, while the use of the Dirichlet-Neumann operator (1.4) in the formulation of the system (1.6), (1.7) is mainly due to Craig, C. Sulem and P.-L. Sulem in [14, 15].

The non-dimensionalized version (1.1) of the water-waves problem, that we use for the dispersive, deep water case relevant in this article, is relying on a more general one, derived by Alvarez-Samaniego and Lannes first in [2] and then presented in more detail in [37], which works out all characteristic parameters of the water wave problem. This is particularly useful for a systematic and analytically reliable derivation of all possible asymptotic limits one may be interested in. Since the ultimate goal of the present article is the justification (see Section 4) of the modulation equations formally derived in Section 3, and since our justification result (Theorem 4.2) follows directly from the stability property of the water-waves problem as presented by Lannes in [37] (see here Theorem 4.1), we chose to study the water-waves problem from the beginning in the form (1.1). This is also the reason for the (at a first glance unusual) notation of the water-depth by $\sqrt{\mu}$. For a full derivation of the water-waves problem in the form (1.1), and an extended and detailed overview of its recent analytical state of the art, we refer the reader to [37].

The water-waves problem (1.1) has the linearization around $(\zeta, \psi) = (0, 0)$

$$(1.9) \quad \begin{cases} \partial_t \zeta - \mathcal{G}[0]\psi = 0, \\ \partial_t \psi + \zeta - \frac{1}{\text{Bo}} \Delta \zeta = 0, \end{cases}$$

with $\mathcal{G}[0]\psi = \partial_z \Phi(\cdot, 0)$, where Φ solves (1.5) with Dirichlet data $\Phi(\cdot, 0) = \psi$ at the surface $\zeta = 0$. Considering the Fourier transform of (1.5) with respect to the horizontal variables $X \in \mathbb{R}^d$, we obtain for each $\xi \in \mathbb{R}^d$ a second-order ODE for $\Phi(\xi, \cdot)$ in the vertical variable z with boundary values at $z = -\sqrt{\mu}$ and $z = 0$, which can be solved uniquely, yielding

$$\widehat{\partial_z \Phi}(\xi, 0) = g_0(\xi) \widehat{\psi}(\xi), \quad g_0(\xi) = |\xi| \tanh(\sqrt{\mu}|\xi|), \quad \xi \in \mathbb{R}^d.$$

Thus, in the Fourier-multiplier notation

$$(1.10) \quad \widehat{f(D)u}(\xi) = f(\xi) \widehat{u}(\xi), \quad \xi \in \mathbb{R}^d, \quad \text{with } D = -i\nabla,$$

we obtain

$$\mathcal{G}[0]\psi = |D| \tanh(\sqrt{\mu}|D|)\psi.$$

Moreover, we obtain that (1.9) allows for plane wave solutions of the form

$$(1.11) \quad \begin{pmatrix} \zeta \\ \psi \end{pmatrix} e^{i(\underline{\xi} \cdot X - \underline{\omega} t)} + \text{c.c.}, \quad \underline{\xi} \in \mathbb{R}^d, \quad \underline{\omega} \in \mathbb{R}, \quad \underline{\zeta}, \underline{\psi} \in \mathbb{C}$$

(with c.c. denoting the complex conjugate of the preceding term(s)), provided the dispersion relation

$$(1.12) \quad \underline{\zeta} = \frac{i\underline{\omega}}{1 + \frac{1}{\text{Bo}}|\underline{\xi}|^2} \underline{\psi} \quad \text{and} \quad \underline{\omega}^2 = \omega^2(\underline{\xi}),$$

is satisfied, with the dispersion function

$$(1.13) \quad \omega(\xi) = \sqrt{(1 + \frac{1}{\text{Bo}}|\xi|^2)g_0(\xi)}, \quad g_0(\xi) = |\xi| \tanh(\sqrt{\mu}|\xi|), \quad \xi \in \mathbb{R}^d.$$

In the case of linear systems one can construct more complicated solutions (wave packets) by superposition of plane waves via Fourier transformation. The analog to this in nonlinear systems is the consideration of modulated plane waves. In the most simple case of amplitude modulation we replace the constants $\underline{\zeta}, \underline{\psi} \in \mathbb{C}$ in (1.11) by slowly varying functions

$$(1.14) \quad \begin{pmatrix} \underline{\zeta}(t', X') \\ \underline{\psi}(t', X') \end{pmatrix} e^{i(\underline{\xi} \cdot X - \underline{\omega} t)} + \text{c.c.}, \quad \underline{\zeta}, \underline{\psi} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C},$$

where $t' = \epsilon t$, $X' = \epsilon X$ with $0 < \epsilon \leq 1$ are new, macroscopic time- and space- variables. The question then is whether the nonlinear system allows approximatively for solutions of this form. By this we mean solutions which *maintain* the above form at least locally with respect to the macroscopic time $t' \leq T$, or, equivalently, for $t \leq T/\epsilon$. Typically, by inserting the two-scale ansatz (1.14) into the nonlinear system, one obtains formally the necessary conditions, viz. the modulation equations, which the macroscopic functions $\underline{\zeta}, \underline{\psi}$ have to satisfy. The modulation equations reveal some qualitative, macroscopic feature in the behavior of the nonlinear system under investigation, which depends of course strongly on the macroscopic scaling used for the modulation. This approach has been used widely in the physics literature for all sorts of dispersive systems. Indeed, one of the oldest fields of application have been water waves, as is exemplified prominently in the work of Whitham [51], to which we refer for a methodical exposition of the ideas behind modulation from the physical point of view.

In nonlinear dispersive systems the central modulation equation is the nonlinear Schrödinger equation (nlS), since it captures the interplay between nonlinearity and dispersion governing the deformation of the envelopes of the wave packets (see e.g. [49] for an overview). For this, the right two-scale ansatz is not (1.14) but rather

$$(1.15) \quad \begin{pmatrix} \underline{\zeta}(t'', X'') \\ \underline{\psi}(t'', X'') \end{pmatrix} e^{i(\underline{\xi} \cdot X - \underline{\omega} t)} + \text{c.c.}, \quad \underline{\zeta}, \underline{\psi} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C},$$

with $t'' = \epsilon t'$, $X'' = X' - \nabla \underline{\omega} t'$, where $\nabla \underline{\omega}$ is the group velocity of the wave packet. Inserting this ansatz (with a corresponding polarization condition for $\underline{\zeta}$) into (1.1), one obtains the nlS equation, which for two-dimensional gravity waves takes the form

$$\partial_t'' \underline{\psi} - i \frac{1}{2} \underline{\omega}'' \partial_x''^2 \underline{\psi} + i c |\underline{\psi}|^2 \underline{\psi} = 0$$

with $c \in \mathbb{R}$ depending on $\underline{\xi}, \underline{\omega}$ and $\partial_t'', \partial_x''$ denoting differentiation with respect to t'', x'' . It was derived by Zakharov [57] for infinite depth and by Hasimoto and Ono [30] for finite depth. In the three-dimensional case of finite depth instead of the nlS one obtains for the scaling (1.15) the Davey-Stewartson (DS) system [16] (see [49] for a detailed discussion of its properties). However, in infinite depth again the nlS is obtained as the modulation equation for the scaling (1.15). Concerning the justification of these modulation equations, this has been achieved for the two-dimensional gravity water-waves problem by Totz and Wu [50] in the case of infinite depth and by Düll, Schneider, and Wayne [18] in the case of finite depth. In the three-dimensional capillary-gravity case there exist consistency results for the nlS

equation [12] and for the DS system [13], where consistency means that the amount by which the approximate solution fails to satisfy the original problem (i.e. the residual) tends to zero in the asymptotic limit with respect to some relevant norm.

In the present article we use the hyperbolic scaling (1.14) and consider three modulated plane waves of that form. We are interested in the modulation equations that govern the macroscopic dynamics of these waves not only in leading order but also with respect to their macroscopic corrections of order $O(\epsilon)$. For the sake of clarity, we first present our exact assumptions, and discuss them afterwards.

We make the two-scale ansatz for approximate solutions of (1.1)

$$(1.16) \quad U_a = \begin{pmatrix} \zeta_a \\ \psi_a \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \psi_0 \end{pmatrix} + \epsilon \begin{pmatrix} \zeta_1 \\ \psi_1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} \zeta_2 \\ \psi_2 \end{pmatrix}$$

with

$$\begin{aligned} \begin{pmatrix} \zeta_0 \\ \psi_0 \end{pmatrix} &= \sum_j \begin{pmatrix} \zeta_{0j} \\ \psi_{0j} \end{pmatrix} e_j + \text{c.c.} + \begin{pmatrix} \zeta_{00} \\ \psi_{00} \end{pmatrix}, \\ \begin{pmatrix} \zeta_1 \\ \psi_1 \end{pmatrix} &= \sum_j \begin{pmatrix} \zeta_{1j} \\ \psi_{1j} \end{pmatrix} e_j + \sum_{ji} \begin{pmatrix} \zeta_{1ji} \\ \psi_{1ji} \end{pmatrix} e_{ji} + \text{c.c.} + \begin{pmatrix} \zeta_{10} \\ \psi_{10} \end{pmatrix}, \\ \begin{pmatrix} \zeta_2 \\ \psi_2 \end{pmatrix} &= \sum_j \begin{pmatrix} \zeta_{2j} \\ \psi_{2j} \end{pmatrix} e_j + \sum_{ji} \begin{pmatrix} \zeta_{2ji} \\ \psi_{2ji} \end{pmatrix} e_{ji} + \sum_{jik} \begin{pmatrix} \zeta_{2jik} \\ \psi_{2jik} \end{pmatrix} e_{jik} + \text{c.c.} + \begin{pmatrix} \zeta_{20} \\ \psi_{20} \end{pmatrix}, \end{aligned}$$

defined according to the following notations and assumptions:

Notation 1.1. (1) All functions $\zeta_{\dots}, \psi_{\dots}$ on the right of $(\zeta_n, \psi_n)^T$, $n = 0, 1, 2$, are complex-valued and depend only on the macroscopic time- and space-variables $0 \leq t' = \epsilon t \leq T$, $X' = \epsilon X = \epsilon(x, y) \in \mathbb{R}^d$ ($d = 1, 2$ and $X = x \in \mathbb{R}$ if $d = 1$), where $0 < \epsilon \leq 1$. Differentiation with respect to t' and X' is denoted by $\partial_{t'}$ and ∇' . The abbreviation c.c. denotes the complex conjugate of all preceding terms.

(2) We introduce the index-sets

$$J = \{1, 2, 3\},$$

$$I = \{(1, 1), (2, 2), (3, 3), (1, \pm 2), (1, \pm 3), (2, \pm 3)\} \supset \{(1, 2), (1, 3), (2, 3)\} = I_<,$$

$$\begin{aligned} K = \{ & (1, 1, 1), (2, 2, 2), (3, 3, 3), \\ & (1, 1, \pm 2), (1, 1, \pm 3), (2, 2, \pm 3), (2, 2, \pm 1), (3, 3, \pm 1), (3, 3, \pm 2), \\ & (1, 2, 3), (1, 2, -3), (1, 3, -2), (2, 3, -1) \}. \end{aligned}$$

We denote summation over these index-sets by

$$\sum_j := \sum_{j \in J}, \quad \sum_{ji} := \sum_{(j,i) \in I}, \quad \sum_{jik} := \sum_{(j,i,k) \in K}.$$

(3a) The functions $e_{\pm j}$ for $j \in J$, e_{ji} for $(j, i) \in I$ and e_{jik} for $(j, i, k) \in K$ are defined through

$$e_{\pm j}(t, X) = e^{\pm i(\xi_j \cdot X - \omega_j t)}, \quad e_{ji} = e_j e_i, \quad e_{jik} = e_j e_i e_k,$$

where the wave-vectors $\xi_j \in \mathbb{R}^d \setminus \{0\}$ and the frequencies $\omega_j = \omega(\xi_j) > 0$ satisfy for each $j \in J$ the dispersion relation $\omega_j^2 = \omega^2(\xi_j)$ with the dispersion function (1.13). We assume that the plane waves e_j , $j \in J$, are mutually different, i.e.

$$(\xi_j, \omega_j) \neq (\xi_i, \omega_i) \quad \forall j, i \in J, j \neq i.$$

Occasionally, we will refer to e_j , e_{ji} and e_{jik} as the first-, second- and third-order harmonics, respectively, and to $1 = e^0$ as the zeroth-order harmonic.

(3b) In analogy to the index-notation for the harmonics, we use the abbreviations

$$\begin{aligned} \xi_{\pm j} &= \pm \xi_j, & \xi_{ji} &= \xi_j + \xi_i, & \xi_{jik} &= \xi_j + \xi_i + \xi_k, \\ \omega_{\pm j} &= \pm \omega_j, & \omega_{ji} &= \omega_j + \omega_i, & \omega_{jik} &= \omega_j + \omega_i + \omega_k, \end{aligned}$$

and

$$\begin{aligned} b_j &= 1 + \frac{1}{\text{Bo}} |\xi_j|^2, & b_{ji} &= 1 + \frac{1}{\text{Bo}} |\xi_{ji}|^2, & b_{jik} &= 1 + \frac{1}{\text{Bo}} |\xi_{jik}|^2, \\ g_j &= g_0(\xi_j), & g_{ji} &= g_0(\xi_{ji}), & g_{jik} &= g_0(\xi_{jik}). \end{aligned}$$

Finally, we denote

$$g'_j = \nabla g_0(\xi_j), \quad g'_{ji} = \nabla g_0(\xi_{ji}), \quad H_j = \frac{1}{2} \nabla' \cdot \mathcal{H}_{g_0}(\xi_j) \nabla',$$

where $\mathcal{H}_{g_0}(\xi)$ is the Hessian matrix of the function g_0 at $\xi \in \mathbb{R}^d$, see (1.13). (Note, in particular, $g_0(0, 0) = 0$, $\nabla g_0(0, 0) = (0, 0)$ and $\mathcal{H}_{g_0}(0, 0) = 2\sqrt{\mu}I$.)

In this notation the following identities hold true:

$$(1.17) \quad 2\omega_j \nabla \omega_j = b_j g'_j + \frac{1}{\text{Bo}} 2\xi_j g_j, \quad \text{where} \quad \nabla \omega_j = \nabla \omega(\xi_j),$$

and

$$(1.18) \quad \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} = \frac{b_j}{\omega_j} H_j \psi_{0j} - \frac{1}{\omega_j} (b_j \nabla_{\text{Bo}} \omega_j \cdot \nabla')^2 \psi_{0j} + \frac{1}{\text{Bo}} \frac{\omega_j}{b_j} \Delta' \psi_{0j},$$

where

$$(1.19) \quad \nabla_{\text{Bo}} \omega_j = \frac{1}{b_j} \left(\nabla \omega_j - \frac{1}{\text{Bo}} 2 \frac{\omega_j}{b_j} \xi_j \right).$$

(3c) The plane waves e_j , $j \in J$, satisfy the non-resonance conditions

$$\omega_{ji}^2 \neq \omega^2(\xi_{ji}) = b_{ji} g_{ji} \quad \forall (j, i) \in I$$

and

$$\omega_{jik}^2 \neq \omega^2(\xi_{jik}) = b_{jik} g_{jik} \quad \forall (j, i, k) \in K.$$

(4) For $u \in H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, see (4.3), of the form

$$u(X) = \sum_{i=1}^k \tilde{u}_i(X') e^{i\xi_i \cdot X}$$

we use the notations $|\tilde{u}|_{H^s} = \sum_{i=1}^k |\tilde{u}_i|_{H^s}$ and

$$u'(X) = \sum_{i=1}^k i\xi_i \tilde{u}_i(\epsilon X) e^{i\xi_i \cdot X}, \quad u''(X) = \sum_{i=1}^k (-|\xi_i|^2) \tilde{u}_i(\epsilon X) e^{i\xi_i \cdot X},$$

such that (with $\Delta' = \nabla' \cdot \nabla'$)

$$(1.20) \quad \nabla u = u' + \epsilon \nabla' u \quad \text{and} \quad \Delta u = u'' + 2\epsilon \nabla' \cdot u' + \epsilon^2 \Delta' u.$$

The motivation for the special form of the ansatz (1.16) is that we want to include in our formal expansion of the capillary-gravity water-waves equation (1.1) the case of quadratic interaction of two modulated plane waves. By quadratic interaction we mean the situation where two such waves generate by multiplication a third plane wave through the (quadratic) resonance of their phases, e.g. $e_1 e_2 = e_3$. In this case one has to consider from the outset all three involved modulated plane waves in order to obtain a closed system of macroscopic equations, and the interaction is manifested macroscopically in leading order, which means that an expansion up to $O(\epsilon)$ -terms in (1.16) would be sufficient.

However, as will be explained below, in the case of pure gravity waves, which is our main focus in the present paper, no such quadratic resonances arise. Such resonances exist only if surface tension is included in the original water-waves equation, see [46] for the two-dimensional case ($d = 1$). Then, naturally, the question arises, whether even in this quadratically non-resonant case any macroscopic coupling can be detected in the next-to-leading-order correction of the leading order amplitudes or in the non-oscillating mean field generated by the waves. Wanting to perform the (unsurprisingly, very cumbersome) formal expansion of the water waves equation for three modulated pulses only once, we chose the ansatz (1.16), which is useful in both cases (quadratically resonant and non-resonant) and for waves with or without surface tension.

As expected, indeed also in the quadratically non-resonant case, the interaction of modulated waves can be traced in the second-order macroscopic system. More precisely, we obtain in Section 3 that in order for the approximation U_a of (1.16) to satisfy formally the water waves equation (1.1) up to residual terms of order $O(\epsilon^3)$, i.e.

$$(1.21) \quad \partial_t U_a + \mathcal{N}_{\epsilon, \sigma}(U_a) = \epsilon^3 (r_2^1, r_2^2)^T,$$

the macroscopic modulation equations

$$(3.17) \quad \begin{cases} \partial_t' \psi_{0j} + \nabla \omega_j \cdot \nabla' \psi_{0j} = 0, \\ \partial_t'^2 \psi_{00} - \sqrt{\mu} \Delta' \psi_{00} = \sum_j \left((g_j^2 - |\xi_j|^2) \partial_t' + 2 \frac{\omega_j}{b_j} \xi_j \cdot \nabla' \right) |\psi_{0j}|^2, \\ \partial_t' \psi_{1j} + \nabla \omega_j \cdot \nabla' \psi_{1j} = E_j \end{cases}$$

with $j \in J = \{1, 2, 3\}$ and

$$(3.9) \quad E_j = i \frac{1}{2} \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} - i \psi_{0j} \left(\frac{b_j}{2\omega_j} (g_j^2 - |\xi_j|^2) \partial_t' + \xi_j \cdot \nabla' \right) \psi_{00} + \tilde{E}_j$$

have to be satisfied, where \tilde{E}_j consists of cubic products of the leading-order amplitudes ψ_{0j} , see (3.10). The other macroscopic functions appearing in U_a can be determined via $\psi_{0j}, \psi_{00}, \psi_{1j}$ or can be chosen arbitrarily.

Of course, more generally one could consider also an arbitrary number of $N \in \mathbb{N}$, $N \geq 3$, modulated plane waves, as was done for other physical settings, e.g. in [23, 22]. In order to keep the presentation more simple and explicit, we chose, however, to consider here only three pulses. Note that the results derived in the present paper can be used in order to obtain the (non-resonant) macroscopic dynamics up to next-to-leading order for two pulses or even for a single pulse, by equating the macroscopic coefficients of the superfluous waves to zero, cf. for two pulses e.g. [29] and for a single pulse [6] (see also Remark 3.3) and [49, §11.1], [37, §8.2.5].

The ansatz (1.16) consists of all first- and higher-order harmonic terms expected to arise up to order ϵ^2 due to the nonlinear nature of the water waves problem, including the non-oscillating terms for the mean field which arise from the interaction of a plane wave with its complex conjugate. Note here, that the index sets I and K of Notation 1.1(2) represent all possibly different second- and third-order harmonics. In this context, we point out that actually we are interested in the weakly nonlinear approximation of solutions to the water-waves problem (1.1) with $\epsilon = 1$, that is to say in an approximation of the form ϵU_a with U_a as in (1.16). However, the expansion of the water waves equation in Section 2 and the derivation of the modulation equations in Section 3 are performed for the equation in the form (1.1) and the ansatz U_a in (1.16).

The reason why the ansatz U_a includes also terms of order ϵ^2 , although we are interested only in an approximation $U_{a,1}$ of up to next-to-leading-order terms of order ϵ (i.e. consisting only of the first two terms of U_a), is that the determining equations for the functions ζ_{1j}, ψ_{1j} arise at order $O(\epsilon^2)$, cf. Corollary 2.5. Moreover, for the justification of the approximation $U_{a,1}$ over time scales of order $O(1/\epsilon)$ we need first to consider an approximation that satisfies (1.1) formally up to residual terms of order $O(\epsilon^3)$, see Section 4 for the details, and [34] for a more general presentation of this approach.

The two non-resonance conditions of Notation 1.1(3c) imply that none of the higher-order harmonic terms are plane waves (or non-oscillating). The first set of conditions is essential for the form of the modulation equations, see (3.17), yielding that in leading order the macroscopic amplitudes ψ_{0j} are just transported with the group velocity $\nabla\omega_j$ of the wave packet, without any macroscopic interaction. In the opposite case, when quadratic resonances appear, one obtains the 'three-wave-interaction equations', a coupled system of three semilinear transport equations for ψ_{0j} containing for each $j \in J$ quadratic products of the other two amplitudes, according to the existing resonances, see [46]. As stated above, for pure gravity water waves this quadratic non-resonance condition is always satisfied. This is known since the 1960s, see [42], while the existence of quadratic resonances in the case of capillary-gravity waves of infinite depth was first proven in [39]. For a more general discussion of resonances of water waves we refer to [28, 46] and the references given therein. For the sake of completeness we give in the following Remark 1.1 a short analytical proof of the non-existence of quadratic resonances for gravity water waves of finite depth.

REMARK 1.1. Assume there are $\xi_1, \xi_2, \xi_1 + \xi_2 \in \mathbb{R}^d \setminus \{0\}$, such that

$$(\omega(\xi_1) \pm \omega(\xi_2))^2 = \omega^2(\xi_1 + \xi_2) \quad \text{with} \quad \omega(\xi) = \sqrt{|\xi| \tanh(\sqrt{\mu}|\xi|)}.$$

Since ω is a radial function, taking square-roots and possibly considering the opposite of some wave-vector ξ_j and relabeling, the equation on the left can always be written in the form

$$\omega(\xi_1) + \omega(\xi_2) = \omega(\xi_1 + \xi_2), \quad \xi_1, \xi_2 \in \mathbb{R}^d \setminus \{0\}.$$

Multiplying by $\mu^{\frac{1}{4}} > 0$ and setting

$$x = \sqrt{\mu}|\xi_1| > 0, \quad \lambda = \frac{|\xi_2|}{|\xi_1|} > 0, \quad c = \frac{\xi_1 \cdot \xi_2}{|\xi_1||\xi_2|} \in [-1, 1],$$

solving this equation for $\xi_1, \xi_2 \in \mathbb{R}^d \setminus \{0\}$ is equivalent to finding roots $(x, \lambda, c) \in (0, \infty) \times (0, \infty) \times [-1, 1]$ of the function

$$r_0(x, \lambda, c) = \sqrt{(1 + \lambda^2 + 2c\lambda) \tanh(x\sqrt{1 + \lambda^2 + 2c\lambda})} - \sqrt{\lambda \tanh(x\lambda)} - \sqrt{\tanh x}.$$

But $r_0(x, \lambda, c) \leq r_0(x, \lambda, 1)$ and

$$r_0(x, \lambda, 1) = \frac{g(x(1+\lambda)) - g(x\lambda)}{\sqrt{x}} - \sqrt{\tanh x}, \quad \text{where } g(y) = \sqrt{y \tanh y},$$

with $r_0(x, 0, 1) = 0$ and

$$\frac{d}{d\lambda} r_0(x, \lambda, 1) = \sqrt{x} (g'(x(1+\lambda)) - g'(x\lambda)) < 0 \quad \forall \lambda \geq 0,$$

the latter due to the strict decreasing of

$$g'(y) = \frac{\tanh y + y(1 - \tanh^2 y)}{2\sqrt{y \tanh y}}, \quad y \geq 0.$$

Hence, we conclude $r_0(x, \lambda, c) < 0$ for all $(x, \lambda, c) \in (0, \infty) \times (0, \infty) \times [-1, 1]$. \square

While the first (quadratic) non-resonance condition of Notation 1.1(3c) is essential for the form of the derived modulation equations, the second (cubic) non-resonance condition is much less so. Indeed, if the first non-resonance condition holds true, cubic resonances do not change the form of the modulation equations, but merely contribute additional cubic products of the leading order amplitudes ψ_{0j} to the source term of one of the equations for the next-to-leading order amplitudes ψ_{1j} , if a third-order harmonic is equal to one of the three considered plane waves. (Note, that in general if plane waves are generated through resonant interaction one always has to include the generated wave in the original ansatz, here (1.16), in order to obtain a closed system of modulation equations.) For the sake of simplicity we do not consider these cases explicitly here, but prefer to impose the cubic non-resonance condition of Notation 1.1(3c) instead. Nevertheless, since the same justification result holds true for these modified modulation equations, the present paper covers completely the justification of the modulation equations up to next-to-leading order for three weakly amplitude-modulated gravity water waves, provided possibly existing cubic resonances generate only one of the three plane waves considered.

We close this introduction by outlining the structure of the article and commenting on its main results. In the following Section 2, after inserting the ansatz (1.16) for the approximation U_a into the water waves equation (1.1), we expand with respect to the steepness parameter $0 < \epsilon \ll 1$ up to residual terms of formal order $O(\epsilon^3)$, for which we give estimates in the Sobolev norms used for the justification of the modulation equations (3.17) in Section 4. The precise formulas for the more involved, though structurally simple, macroscopic coefficients are given in an Appendix. Thus, as a byproduct, we provide a complete formal explicit expansion including all terms of order ϵ^2 for three amplitude modulated plane waves with the hyperbolic scaling $t' = \epsilon t$, $X' = \epsilon X$ for the capillary-gravity water waves problem of finite depth, that can be used independently, also for only one or two waves.

Then, in Section 3, we derive the necessary conditions on the macroscopic coefficients of U_a in order for the latter to satisfy (1.1) up to the residual terms of

order $O(\epsilon^3)$, see (1.21). In particular we obtain the modulation equations (3.17). Finally, in Section 4 we justify the derived modulation equations as a macroscopic limit to the gravity water waves problem (1.1) with $\epsilon = 1$, viz.

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0, \\ \partial_t \psi + \zeta + \frac{1}{2}|\nabla \psi|^2 - \frac{(\mathcal{G}[\zeta]\psi + \nabla \zeta \cdot \nabla \psi)^2}{2(1 + |\nabla \zeta|^2)} = 0, \end{cases}$$

over the macroscopic time T/ϵ , that is to say, we show that the approximation $\epsilon U_{a,1}$, which consists of only the first two terms on the right hand side of (1.16), with the macroscopic functions given through the solutions of (3.17) up to the time $T > 0$, maintain a distance of order $\epsilon^{3-d/2}$ to the solution U of the original problem over this time interval and with respect to a suitable Sobolev norm, if their distance is of this order at the initial time $t = 0$ (for the precise result, see Theorem 4.2). Here, the reduction of the order, compared to the formal one, is due to the scaling of the macroscopic time and space variables. Note, that the approximation $\epsilon U_{a,1}$ contains terms of order ϵ and ϵ^2 , and hence the obtained result is clearly more valuable in the one-dimensional case $d = 1$, fully justifying the macroscopic interaction in next-to-leading order for three weakly modulated gravity water waves. In order to improve the result in the case $d = 2$, one could try to adapt methods used in nonlinear optics (see, e.g., [27, 9], as pointed out in [37, fn. 10, p. 232]), which is left open here for future consideration.

As already mentioned, our justification result of Theorem 4.2 is in principle an application of the well-posedness result of Alvarez-Samaniego and Lannes [36, 2, 37] on gravity water waves of finite depth for times of order $O(1/\epsilon)$, which is exactly the hyperbolic time-scale of the macroscopic limit considered in the present paper. This result was extended to the case of two-fluid interfaces with surface tension in [38], which contains as a special case capillary-gravity water waves, see also [37, Ch. 9]. However, in this case the energy norm used includes also higher-order time derivatives. Thus, we chose to treat in the present paper only the justification of the macroscopic interaction equations (3.17) in the case without surface tension, postponing to future work the treatment of the capillary-gravity case, which anyway allows also for quadratic resonances, as explained above. Note, that in the resonant, one-dimensional case $d = 1$ of finite depth, the leading-order macroscopic 'three-wave interaction equations' have been justified by Schneider and Wayne in [46], using Lagrangian coordinates. Moreover, it is expected that an analogous approach as the one presented here, can be performed also for the case of infinite depth.

Concluding, we would like to mention that the macroscopic limit (3.17) derived and justified in the present paper is an alternative to a three-wave generalization of the Benney-Roskes system [6], see Remark 3.3. The latter is the relevant one with respect to the longer dispersive time-scale $t'' = \epsilon t' = \epsilon^2 t$. However, on such a long time-scale there exist no well-posedness results up to now, neither for the original water-waves problem of finite depth nor for the Benney-Roskes system, in contrast to the situation here, where both the original and the derived models are well-posed on the relevant hyperbolic time-scale $t' = \epsilon t$.

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2. TWO-SCALE EXPANSION AND ESTIMATES FOR THE RESIDUALS

The formal derivation of the modulation equations consists in plugging the ansatz (1.16) into the original water waves problem (1.1), expanding with respect to ϵ , and equating all terms up to order ϵ^2 to 0. In the present section we perform the first two steps by writing out the terms up to order ϵ^2 and giving estimates with respect to the $H^s(\mathbb{R}^d)$ -norm $|\cdot|_{H^s}$, see (4.3), for the residual terms of formal order $O(\epsilon^3)$. The third step, i.e. the actual derivation of the modulation equations, is performed in the next section.

Concerning the time- and space-derivatives of the approximation $U_a = (\zeta_a, \psi_a)^T$ in (1.16), we obtain immediately their expansions with respect to ϵ , viz.

$$\begin{aligned}
 (2.1) \quad \partial_t \zeta_a &= \sum_j (-i\omega_j) \zeta_{0j} e_j + \text{c.c.} \\
 &+ \epsilon \left(\sum_j (\partial'_t \zeta_{0j} - i\omega_j \zeta_{1j}) e_j + \sum_{ji} (-i\omega_{ji}) \zeta_{1ji} e_{ji} + \text{c.c.} + \partial'_t \zeta_{00} \right) \\
 &+ \epsilon^2 \left(\sum_j (\partial'_t \zeta_{1j} - i\omega_j \zeta_{2j}) e_j + \sum_{ji} (\partial'_t \zeta_{1ji} - i\omega_{ji} \zeta_{2ji}) e_{ji} \right. \\
 &\quad \left. + \sum_{jik} (-i\omega_{jik}) \zeta_{2jik} e_{jik} + \text{c.c.} + \partial'_t \zeta_{10} \right) \\
 &+ \epsilon^3 \left(\sum_j \partial'_t \zeta_{2j} e_j + \sum_{ji} \partial'_t \zeta_{2ji} e_{ji} + \sum_{jik} \partial'_t \zeta_{2jik} e_{jik} + \text{c.c.} + \partial'_t \zeta_{20} \right),
 \end{aligned}$$

where we recall that ∂'_t denotes differentiation with respect to $t' = \epsilon t$, and

$$\begin{aligned}
 \nabla \zeta_a &= \sum_j i\xi_j \zeta_{0j} e_j + \text{c.c.} \\
 &+ \epsilon \left(\sum_j (\nabla' \zeta_{0j} + i\xi_j \zeta_{1j}) e_j + \sum_{ji} i\xi_{ji} \zeta_{1ji} e_{ji} + \text{c.c.} + \nabla' \zeta_{00} \right) \\
 &+ \epsilon^2 \left(\sum_j (\nabla' \zeta_{1j} + i\xi_j \zeta_{2j}) e_j + \sum_{ji} (\nabla' \zeta_{1ji} + i\xi_{ji} \zeta_{2ji}) e_{ji} \right. \\
 &\quad \left. + \sum_{jik} i\xi_{jik} \zeta_{2jik} e_{jik} + \text{c.c.} + \nabla' \zeta_{10} \right) \\
 &+ \epsilon^3 \left(\sum_j \nabla' \zeta_{2j} e_j + \sum_{ji} \nabla' \zeta_{2ji} e_{ji} + \sum_{jik} \nabla' \zeta_{2jik} e_{jik} + \text{c.c.} + \nabla' \zeta_{20} \right),
 \end{aligned}$$

where ∇' denotes differentiation with respect to $X' = \epsilon X = \epsilon(x, y)$. Analogous expansions hold true for $\partial_t \psi_a$ and $\nabla \psi_a$.

In order to obtain an expansion in terms of ϵ for $\mathcal{N}_{\epsilon, \sigma}(U_a)$ in (1.1), given through (1.2), (1.3), we obviously need first of all an expansion of $\mathcal{G}[\epsilon \zeta_a] \psi_a$, defined by (1.4). For this, we can rely on the expansions of the Dirichlet-Neumann operator given in [37, Lemmata 8.11, 8.12], which we adapt to the present situation.

Proposition 2.1. (1) *Let ζ, ψ be of the form in Notation 1.1.(4) with $\tilde{\zeta}_i \in H^{s+1+t_0}(\mathbb{R}^d)$, $\tilde{\psi}_i \in H^{s+1}(\mathbb{R}^d)$, where $s \geq 0$, $t_0 > d/2$, and*

$$1 - \epsilon |\zeta|_\infty \geq h_{\min} > 0, \quad 0 < \epsilon \leq 1.$$

Then, for $\mathcal{G}[\epsilon\zeta]\psi$ given in (1.4) with $1 \leq \mu \leq \mu_{\max} < \infty$, we have

$$(2.2) \quad \mathcal{G}[\epsilon\zeta]\psi = \mathcal{G}_0\psi + \sum_{m=1}^n \epsilon^m \mathcal{G}_m[\zeta]\psi + \epsilon^{n+1} \mathcal{R}_n[\zeta]\psi, \quad n = 0, 1, 2,$$

with

$$\begin{aligned} \mathcal{G}_0\psi &= \mathcal{G}[0]\psi = |D| \tanh(\sqrt{\mu}|D|)\psi, \\ \mathcal{G}_1[\zeta]\psi &= -\mathcal{G}_0(\zeta \mathcal{G}_0\psi) - \nabla \cdot (\zeta \nabla \psi), \\ \mathcal{G}_2[\zeta]\psi &= \mathcal{G}_0(\zeta \mathcal{G}_0(\zeta \mathcal{G}_0\psi)) + \frac{1}{2} \Delta(\zeta^2 \mathcal{G}_0\psi) + \frac{1}{2} \mathcal{G}_0(\zeta^2 \Delta \psi) \end{aligned}$$

and

$$(2.3) \quad |\mathcal{G}[\epsilon\zeta]\psi|_{H^s} \leq \epsilon^{-d/2} M(s \vee t_0 + 1, \tilde{\zeta}) |\tilde{\psi}|_{H^{s+1}},$$

$$(2.4) \quad |\mathcal{R}_n[\zeta]\psi|_{H^s} \leq \epsilon^{-d/2} M(s + 1 + t_0, \tilde{\zeta}) |\tilde{\psi}|_{H^{s+1}},$$

where $s \vee t_0 = \max\{s, t_0\}$ and

$$M(s, \tilde{\zeta}) = C(h_{\min}^{-1}, \mu_{\max}, |\xi_i|, |\tilde{\zeta}|_{H^s})$$

is a nondecreasing function of each of its arguments.

- (2) Let u be as in Notation 1.1.(4) with $\tilde{u}_i \in H^{s+n+1}(\mathbb{R}^d)$ and \mathcal{G}_0 as above. Then, with the Notation 1.1.(3b), we have

$$(2.5) \quad \mathcal{G}_0 u = \sum_{m=0}^n \epsilon^m G_m u + \epsilon^{n+1} R_n u, \quad n = 0, 1, 2,$$

with

$$\begin{aligned} G_0 u(X) &= \sum_{i=1}^k g_0(\xi_i) \tilde{u}_i(X') e^{i\xi_i \cdot X}, \\ G_1 u(X) &= -i \sum_{i=1}^k \nabla g_0(\xi_i) \cdot \nabla' \tilde{u}_i(X') e^{i\xi_i \cdot X}, \\ G_2 u(X) &= -\frac{1}{2} \sum_{i=1}^k \nabla' \cdot \mathcal{H}_{g_0}(\xi_i) \nabla' \tilde{u}_i(X') e^{i\xi_i \cdot X} \end{aligned}$$

and

$$|R_n u|_{H^s} \leq \epsilon^{-d/2} C(\mu_{\max}, |\xi_i|) |\tilde{u}|_{H^{s+n+1}}.$$

Proof. The estimate (2.3) is obtained through the identification

$$(2.6) \quad \mathcal{G}[\epsilon\zeta]\psi = \frac{1}{\sqrt{\mu}} \mathcal{G}_{\mu,1} \left[\frac{1}{\sqrt{\mu}} \epsilon\zeta, 0 \right] \psi, \quad 1 \leq \mu \leq \mu_{\max} < \infty,$$

from [37, Theorem 3.15 (1)] and (4.17), in the form

$$(2.7) \quad |\mathcal{G}[\epsilon\zeta]\psi|_{H^s} \leq C(h_{\min}^{-1}, \mu_{\max}, |\epsilon\zeta|_{H^{s \vee t_0+1}}) |\nabla \psi|_{H^s},$$

together with the estimate

$$(2.8) \quad |u|_{H^s} \leq C(|\xi_i|) \epsilon^{-d/2} |\tilde{u}|_{H^s}, \quad s \geq 0, \quad 0 < \epsilon \leq 1,$$

for functions as in Notation 1.1.(4), exploiting $d = 1, 2$.

Similarly, the expansion (2.2) (based on a Taylor-expansion of $\mathcal{G}[\epsilon\zeta]\psi$ around $\zeta = 0$ in the direction ζ and on the analyticity of the Dirichlet-Neumann operator) and (2.4) follow by (2.6) and (2.8) from [37, Proposition 3.44 (for $k = 1$)], see also [37, Remark 3.47 and Lemma 8.11]. We require here a higher regularity of $\tilde{\zeta}$, in

line with [12, 13] (see, in particular, [13, Theorem 4.7] for $s \in \mathbb{N}_0$ and $d = 2$), in order to obtain (2.4) in a more straightforward manner.

The second point follows from [37, Lemma 8.12], see also [12, 13]. \square

We use the previous proposition in order to expand $\mathcal{N}_{\epsilon, \sigma}(U_a)$ with respect to ϵ , writing out explicitly the terms of orders up to ϵ^2 and providing H^s -norm estimates for the residual terms of formal order $O(\epsilon^3)$.

Corollary 2.2. *For $\mathcal{N}_{\epsilon, \sigma}(U_a)$ of (1.2), (1.3), (1.16), the notations of Notation 1.1 and Proposition 2.1 with $t_0 = 3/2$, $s \geq 1$, and \mathfrak{P} as in (4.2), we have*

$$\begin{aligned} \mathcal{G}[\epsilon \zeta_a] \psi_a &= G_0 \psi_0 \\ &+ \epsilon (G_1 \psi_0 + G_0 \psi_1 - G_0(\zeta_0 G_0 \psi_0) - \zeta'_0 \cdot \psi'_0 - \zeta_0 \psi''_0) \\ &+ \epsilon^2 (G_2 \psi_0 + G_1 \psi_1 + G_0 \psi_2 - G_1(\zeta_0 G_0 \psi_0) - G_0(\zeta_0 G_1 \psi_0) \\ &\quad - G_0(\zeta_1 G_0 \psi_0) - G_0(\zeta_0 G_0 \psi_1) + G_0(\zeta_0 G_0(\zeta_0 G_0 \psi_0)) \\ &\quad - \zeta'_0 \cdot \nabla' \psi_0 - \nabla' \zeta_0 \cdot \psi'_0 - 2\zeta_0 \nabla' \cdot \psi'_0 \\ &\quad - \zeta'_1 \cdot \psi'_0 - \zeta_1 \psi''_0 - \zeta'_0 \cdot \psi'_1 - \zeta_0 \psi''_1 + \tfrac{1}{2}(\zeta_0^2 G_0 \psi_0)'' + \tfrac{1}{2}G_0(\zeta_0^2 \psi''_0)) \\ &+ \epsilon^3 R_2^1 \end{aligned}$$

with

$$(2.9) \quad |R_2^1|_{H^s} \leq \epsilon^{-d/2} M(s + 5/2, \tilde{\zeta}_a) (|\tilde{\psi}_0|_{H^{s+3}} + |\tilde{\psi}_1|_{H^{s+2}} + |\tilde{\psi}_2|_{H^{s+1}})$$

and

$$\begin{aligned} -\mathcal{N}_{\epsilon, \sigma}^2(U_a) &= -\zeta_0 + \tfrac{1}{\text{Bo}} \zeta''_0 \\ &+ \epsilon \left(-\zeta_1 - \tfrac{1}{2} |\psi'_0|^2 + \tfrac{1}{2} (G_0 \psi_0)^2 + \tfrac{1}{\text{Bo}} (2\nabla' \cdot \zeta'_0 + \zeta''_1) \right) \\ &+ \epsilon^2 \left(-\zeta_2 - \psi'_0 \cdot (\nabla' \psi_0 + \psi'_1) \right. \\ &\quad \left. + (G_1 \psi_0 + G_0 \psi_1 - G_0(\zeta_0 G_0 \psi_0) - \zeta_0 \psi''_0) G_0 \psi_0 \right. \\ &\quad \left. + \tfrac{1}{\text{Bo}} (\Delta' \zeta_0 + 2\nabla' \cdot \zeta'_1 + \zeta''_2 - \tfrac{1}{2} (|\zeta'_0|^2 \zeta'_0)') \right) \\ &+ \epsilon^3 R_2^2 \end{aligned}$$

with

$$(2.10) \quad |\mathfrak{P} R_2^2|_{H^s} \leq \epsilon^{-d/2} M(s + 3, \tilde{\zeta}_a) \left(C(|\tilde{\psi}_0|_{H^{s+7/2}}, |\tilde{\psi}_1|_{H^{s+5/2}}, |\tilde{\psi}_2|_{H^{s+3/2}}) + \tfrac{1}{\text{Bo}} \right).$$

Proof. We start with the expansion of $\mathcal{G}[\epsilon \zeta_a] \psi_a$. First, we use (2.2) and obtain

$$\begin{aligned} \mathcal{G}[\epsilon \zeta_a] \psi_a &= \mathcal{G}_0 \psi_0 + \epsilon (\mathcal{G}_0 \psi_1 - \mathcal{G}_0(\zeta_0 \mathcal{G}_0 \psi_0) - \nabla \cdot (\zeta_0 \nabla \psi_0)) \\ &+ \epsilon^2 (\mathcal{G}_0 \psi_2 - \mathcal{G}_0(\zeta_1 \mathcal{G}_0 \psi_0) - \mathcal{G}_0(\zeta_0 \mathcal{G}_0 \psi_1) - \nabla \cdot (\zeta_1 \nabla \psi_0 + \zeta_0 \nabla \psi_1) \\ &\quad + \mathcal{G}_0(\zeta_0 \mathcal{G}_0(\zeta_0 \mathcal{G}_0 \psi_0)) + \tfrac{1}{2} \Delta(\zeta_0^2 \mathcal{G}_0 \psi_0) + \tfrac{1}{2} \mathcal{G}_0(\zeta_0^2 \Delta \psi_0)) \\ &+ \epsilon^3 \mathcal{R}_2^a \end{aligned}$$

with

$$\begin{aligned} \mathcal{R}_2^a &= \mathcal{R}_0[\zeta_a] \psi_2 + \mathcal{R}_1[\zeta_a] \psi_1 + \mathcal{R}_2[\zeta_a] \psi_0 + \mathcal{G}_1[\zeta_1 + \epsilon \zeta_2] \psi_1 + \mathcal{G}_1[\zeta_2] \psi_0 \\ &+ \mathcal{G}_0((\zeta_1 + \epsilon \zeta_2) \mathcal{G}_0(\zeta_0 \mathcal{G}_0 \psi_0)) + \mathcal{G}_0(\zeta_a \mathcal{G}_0((\zeta_1 + \epsilon \zeta_2) \mathcal{G}_0 \psi_0)) \\ &+ \tfrac{1}{2} \Delta((\zeta_1 + \epsilon \zeta_2)(\zeta_0 + \zeta_a) \mathcal{G}_0 \psi_0) + \tfrac{1}{2} \mathcal{G}_0((\zeta_1 + \epsilon \zeta_2)(\zeta_0 + \zeta_a) \Delta \psi_0). \end{aligned}$$

Then, we expand \mathcal{G}_0 according to (2.5), getting

$$\begin{aligned}\mathcal{G}[\epsilon\zeta_a]\psi_a &= G_0\psi_0 + \epsilon(G_1\psi_0 + G_0\psi_1 - G_0(\zeta_0 G_0\psi_0) - \nabla \cdot (\zeta_0 \nabla \psi_0)) \\ &\quad + \epsilon^2(G_2\psi_0 + G_1\psi_1 + G_0\psi_2 - G_1(\zeta_0 G_0\psi_0) - G_0(\zeta_0 G_1\psi_0) \\ &\quad - G_0(\zeta_1 G_0\psi_0) - G_0(\zeta_0 G_0\psi_1) + G_0(\zeta_0 G_0(\zeta_0 G_0\psi_0)) \\ &\quad - \nabla \cdot (\zeta_1 \nabla \psi_0 + \zeta_0 \nabla \psi_1) + \frac{1}{2}\Delta(\zeta_0^2 G_0\psi_0) + \frac{1}{2}G_0(\zeta_0^2 \Delta\psi_0)) \\ &\quad + \epsilon^3 R_2^a + \epsilon^3 \mathcal{R}_2^a\end{aligned}$$

with

$$\begin{aligned}R_2^a &= R_2\psi_0 + R_1\psi_1 + R_0\psi_2 - R_1(\zeta_0 G_0\psi_0) - R_0(\zeta_0 G_1\psi_0) - \mathcal{G}_0(\zeta_0 R_1\psi_0) \\ &\quad - R_0(\zeta_1 G_0\psi_0) - \mathcal{G}_0(\zeta_1 R_0\psi_0) - R_0(\zeta_0 G_0\psi_1) - \mathcal{G}_0(\zeta_0 R_0\psi_1) \\ &\quad + R_0(\zeta_0 G_0(\zeta_0 G_0\psi_0)) + \mathcal{G}_0(\zeta_0 R_0(\zeta_0 G_0\psi_0)) + \mathcal{G}_0(\zeta_0 \mathcal{G}_0(\zeta_0 R_0\psi_0)) \\ &\quad + \frac{1}{2}\Delta(\zeta_0^2 R_0\psi_0) + \frac{1}{2}R_0(\zeta_0^2 \Delta\psi_0).\end{aligned}$$

Finally, we expand the ∇ - and Δ -operators according to (1.20), obtaining

$$\begin{aligned}& -\epsilon \nabla \cdot (\zeta_0 \nabla \psi_0) + \epsilon^2(-\nabla \cdot (\zeta_1 \nabla \psi_0 + \zeta_0 \nabla \psi_1) + \frac{1}{2}\Delta(\zeta_0^2 G_0\psi_0) + \frac{1}{2}G_0(\zeta_0^2 \Delta\psi_0)) \\ &= -\epsilon(\zeta_0' \cdot \psi_0' + \zeta_0 \psi_0'') - \epsilon^2(\zeta_0' \cdot \nabla' \psi_0 + \nabla' \zeta_0 \cdot \psi_0' + 2\zeta_0 \nabla' \cdot \psi_0') \\ &\quad + \epsilon^2(-\zeta_1' \cdot \psi_0' - \zeta_1 \psi_0'' - \zeta_0' \cdot \psi_1' - \zeta_0 \psi_1'' + \frac{1}{2}(\zeta_0^2 G_0\psi_0)'' + \frac{1}{2}G_0(\zeta_0^2 \psi_0'')) + \epsilon^3 P_2^a\end{aligned}$$

with

$$\begin{aligned}P_2^a &= -(\nabla' \zeta_0 \cdot \nabla' \psi_0 + \zeta_0 \Delta' \psi_0) \\ &\quad - (\nabla' \zeta_1 \cdot \nabla \psi_0 + \zeta_1' \cdot \nabla' \psi_0 + \zeta_1(2\nabla' \cdot \psi_0' + \epsilon \Delta' \psi_0)) \\ &\quad - (\nabla' \zeta_0 \cdot \nabla \psi_1 + \zeta_0' \cdot \nabla' \psi_1 + \zeta_0(2\nabla' \cdot \psi_1' + \epsilon \Delta' \psi_1)) \\ &\quad + \nabla' \cdot (\zeta_0^2 G_0\psi_0)' + \epsilon \frac{1}{2}\Delta'(\zeta_0^2 G_0\psi_0) + G_0(\zeta_0^2 \nabla' \cdot \psi_0') + \epsilon \frac{1}{2}G_0(\zeta_0^2 \Delta' \psi_0).\end{aligned}$$

Altogether, we obtain $\mathcal{G}[\epsilon\zeta_a]\psi_a$ as in the statement of the corollary, with

$$R_2^1 = P_2^a + R_2^a + \mathcal{R}_2^a,$$

for which we obtain the estimate (2.9), by using the estimates (and expansions) of Proposition 2.1, the product estimates of Lemma 2.3 below, and (2.8).

Analogously, the (shorter) first-order approximation of $\mathcal{G}[\epsilon\zeta_a]\psi_a$ reads

$$\mathcal{G}[\epsilon\zeta_a]\psi_a = G_0\psi_0 + \epsilon(G_1\psi_0 + G_0\psi_1 - G_0(\zeta_0 G_0\psi_0) - \zeta_0' \cdot \psi_0' - \zeta_0 \psi_0'') + \epsilon^2 R_1^1$$

with

$$\begin{aligned}(2.11) \quad R_1^1 &= \mathcal{G}[\epsilon\zeta_a]\psi_2 + \mathcal{R}_0[\zeta_a]\psi_1 + \mathcal{R}_1[\zeta_a]\psi_0 + \mathcal{G}_1[\zeta_1 + \epsilon\zeta_2]\psi_0 \\ &\quad + R_1\psi_0 + R_0\psi_1 - R_0(\zeta_0 G_0\psi_0) - \mathcal{G}_0(\zeta_0 R_0\psi_0) \\ &\quad - (\zeta_0' \cdot \nabla' \psi_0 + \nabla' \zeta_0 \cdot \nabla \psi_0 + 2\zeta_0 \nabla' \cdot \psi_0' + \epsilon \zeta_0 \Delta' \psi_0),\end{aligned}$$

from which we obtain as above

$$(2.12) \quad |R_1^1|_{H^s} \leq \epsilon^{-d/2} M(s+5/2, \tilde{\zeta}_a)(|\tilde{\psi}_0|_{H^{s+2}} + |\tilde{\psi}_1|_{H^{s+1}} + |\tilde{\psi}_2|_{H^{s+1}});$$

while the zeroth-order approximation of $\mathcal{G}[\epsilon\zeta_a]\psi_a$ is given by

$$\mathcal{G}[\epsilon\zeta_a]\psi_a = G_0\psi_0 + \epsilon R_0^1$$

with

$$R_0^1 = \mathcal{G}[\epsilon\zeta_a](\psi_1 + \epsilon\psi_2) + \mathcal{R}_0[\zeta_a]\psi_0 + R_0\psi_0,$$

which leads to

$$(2.13) \quad |R_0^1|_{H^s} \leq \epsilon^{-d/2} M(s + 5/2, \tilde{\zeta}_a) |\tilde{\psi}_a|_{H^{s+1}}.$$

We turn now to the expansion of $\mathcal{N}_{\epsilon,\sigma}^2(U_a)$. First, we have

$$|\nabla \psi_a|^2 = |\psi'_0|^2 + \epsilon 2\psi'_0 \cdot (\nabla' \psi_0 + \psi'_1) + \epsilon^2 \rho_1^a$$

with

$$\rho_1^a = 2\psi'_0 \cdot (\nabla' \psi_1 + \nabla \psi_2) + |\nabla' \psi_0 + \nabla(\psi_1 + \epsilon \psi_2)|^2.$$

Then, using the first- and zeroth-order expansions of $\mathcal{G}[\epsilon \zeta_a] \psi_a$ above, we obtain

$$\begin{aligned} & \frac{(\mathcal{G}[\epsilon \zeta_a] \psi_a + \epsilon \nabla \zeta_a \cdot \nabla \psi_a)^2}{1 + \epsilon^2 |\nabla \zeta_a|^2} \\ &= (G_0 \psi_0)^2 + 2\epsilon (G_1 \psi_0 + G_0 \psi_1 - G_0(\zeta_0 G_0 \psi_0) - \zeta_0 \psi''_0) G_0 \psi_0 + \epsilon^2 r_1^a \end{aligned}$$

with

$$\begin{aligned} r_1^a &= 2[R_1^1 + \zeta'_0 \cdot (\nabla' \psi_0 + \nabla(\psi_1 + \epsilon \psi_2)) + (\nabla' \zeta_0 + \nabla(\zeta_1 + \epsilon \zeta_2)) \cdot \nabla \psi_a] G_0 \psi_0 \\ &+ (R_0^1 + \nabla \zeta_a \cdot \nabla \psi_a)^2 - (\mathcal{G}[\epsilon \zeta_a] \psi_a + \epsilon \nabla \zeta_a \cdot \nabla \psi_a)^2 \frac{|\nabla \zeta_a|^2}{1 + \epsilon^2 |\nabla \zeta_a|^2}. \end{aligned}$$

Finally, we get

$$\begin{aligned} \nabla \cdot \left(\frac{\nabla \zeta_a}{\sqrt{1 + \epsilon^2 |\nabla \zeta_a|^2}} \right) &= \zeta''_0 + \epsilon(2\nabla' \cdot \zeta'_0 + \zeta''_1) \\ &+ \epsilon^2 (\Delta' \zeta_0 + 2\nabla' \cdot \zeta'_1 + \zeta''_2 - \frac{1}{2}(|\zeta'_0|^2 \zeta'_0)') + \epsilon^3 s_2^a \end{aligned}$$

with

$$\begin{aligned} s_2^a &= \Delta' \zeta_1 + \nabla' \cdot \zeta'_2 + \nabla \cdot \nabla' \zeta_2 - \frac{1}{2} \nabla' \cdot (|\zeta'_0|^2 \zeta'_0) \\ &- \frac{1}{2} \nabla \cdot [((\nabla' \zeta_0 + \nabla(\zeta_1 + \epsilon \zeta_2)) \cdot (\zeta'_0 + \nabla \zeta_a)) \zeta'_0 + |\nabla \zeta_a|^2 (\nabla' \zeta_0 + \nabla(\zeta_1 + \epsilon \zeta_2))] \\ &+ \epsilon \nabla \cdot \left(\frac{(4 + r^2(1 + r)) |\nabla \zeta_a|^4 \nabla \zeta_a}{2r(1 + r)(2 + r(1 + r^2))} \right), \quad \text{where } r = \sqrt{1 + \epsilon^2 |\nabla \zeta_a|^2}. \end{aligned}$$

Summarizing, we get the expansion of $\mathcal{N}_{\epsilon,\sigma}^2(U_a)$ presented in the statement, with

$$R_2^2 = -\frac{1}{2} \rho_1^a + \frac{1}{2} r_1^a + \frac{1}{\text{Bo}} s_2^a.$$

Using again Proposition 2.1, Lemma 2.3 below, (2.8), and the estimates (2.12), (2.13) with (4.17), we obtain for $s \geq 1$ the estimate (2.10). Note, that we estimated $|r_1^a|_{H^{s+1/2}}$, by expanding $R_0^1, R_1^1, \mathcal{G}[\epsilon \zeta_a] \psi_a$ up to residual terms of order $O(\epsilon)$, in order to control the appearing products with respect to ϵ . This leads to an increase by one order of the regularity required for $\tilde{\psi}_a$.

For future use, we present also the first-order expansion of $\mathcal{N}_{\epsilon,\sigma}^2(U_a)$

$$\begin{aligned} -\mathcal{N}_{\epsilon,\sigma}^2(U_a) &= -\zeta_0 + \frac{1}{\text{Bo}} \zeta''_0 + \epsilon \left(-\zeta_1 - \frac{1}{2} |\psi'_0|^2 + \frac{1}{2} (G_0 \psi_0)^2 + \frac{1}{\text{Bo}} (2\nabla' \cdot \zeta'_0 + \zeta''_1) \right) \\ &+ \epsilon^2 R_1^2 \end{aligned}$$

with

$$(2.14) \quad R_1^2 = -\zeta_2 - \frac{1}{2} \rho_0^a + \frac{1}{2} r_0^a + \frac{1}{\text{Bo}} s_1^a,$$

where

$$\begin{aligned}\rho_0^a &= (\psi'_0 + \nabla \psi_a) \cdot (\nabla' \psi_0 + \nabla(\psi_1 + \epsilon \psi_2)), \\ r_0^a &= (R_0^1 + \nabla \zeta_a \cdot \nabla \psi_a)(G_0 \psi_0 + \mathcal{G}[\epsilon \zeta_a] \psi_a + \epsilon \nabla \zeta_a \cdot \nabla \psi_a) \\ &\quad - \epsilon (\mathcal{G}[\epsilon \zeta_a] \psi_a + \epsilon \nabla \zeta_a \cdot \nabla \psi_a)^2 \frac{|\nabla \zeta_a|^2}{1 + \epsilon^2 |\nabla \zeta_a|^2}, \\ s_1^a &= \Delta' \zeta_0 + \nabla' \cdot \zeta_1' + \nabla \cdot \nabla' \zeta_1 + \Delta \zeta_2 - \nabla \cdot \left(\frac{|\nabla \zeta_a|^2 \nabla \zeta_a}{r(1+r)} \right), \quad r = \sqrt{1 + \epsilon^2 |\nabla \zeta_a|^2},\end{aligned}$$

which leads to

$$(2.15) \quad |\mathfrak{P} R_1^2|_{H^s} \leq \epsilon^{-d/2} M(s+3, \tilde{\zeta}_a) \left(C(|\tilde{\psi}_0|_{H^{s+5/2}}, |\tilde{\psi}_1 + \epsilon \tilde{\psi}_2|_{H^{s+3/2}}) + \frac{1}{\text{Bo}} \right).$$

□

In the following lemma we list the product estimates for functions in $H^s(\mathbb{R}^d)$ that we used in order to obtain the estimates (2.9) – (2.15). For their proof we refer to [37, Appendix B.1.1] and [2, (2.2)], and the references given therein. Recall also that $H^{t_0}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $t_0 > d/2$. We use the notation

$$A_s + \langle B_s \rangle_{s>r} = \begin{cases} A_s & \text{if } s \leq r, \\ A_s + B_s & \text{if } s > r. \end{cases}$$

Lemma 2.3. *Let $t_0 > d/2$. Then, for $f, g \in H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, the following estimates hold true:*

- (1) *if $s_1, s_2 \in \mathbb{R}$, $s_1, s_2 \geq s$, $0 \vee (s+t_0) \leq s_1 + s_2$, then $|fg|_{H^s} \leq C|f|_{H^{s_1}}|g|_{H^{s_2}}$;*
- (2) *if $s \geq 0$, then $|fg|_{H^s} \leq C(|f|_\infty|g|_{H^s} + |f|_{H^s}|g|_\infty)$;*
- (3) *if $s \geq 0$ and $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^m)$, $F(0) = 0$, then $|F(u)|_{H^s} \leq C(|u|_\infty)|u|_{H^s}$;*
- (4) *if $s \geq -t_0$ and $1 + g(X) \geq c_0 > 0 \forall X \in \mathbb{R}^d$, then*

$$\left| \frac{f}{1+g} \right|_{H^s} \leq C\left(\frac{1}{c_0}, |g|_{H^{t_0}}\right)(|f|_{H^s} + \langle |f|_{H^{t_0}} |g|_{H^s} \rangle_{s>t_0}).$$

We can now insert into the expansions of $\mathcal{G}[\epsilon \zeta_a] \psi_a$ and $\mathcal{N}_{\epsilon, \sigma}^2(U_a)$ with respect to ϵ (Corollary 2.2) the decompositions of the $(\zeta_n, \psi_n)^T$ -terms into their harmonics, see after (1.16), and expand at each order of ϵ with respect to the different harmonics. Due to the nonlinearity of the terms of the ϵ -expansions and the fact that the inserted terms already contain higher-order harmonics, which moreover result from three different carrier waves, this leads to very involved formulas. The following proposition gives the relevant structure of these expansions containing the information needed for the derivation and characterization of the modulation equations, and we refer to the Appendix for exact formulas of the more involved expressions.

Proposition 2.4. *With Notation 1.1 and $(\zeta_a, \psi_a)^T$ as in (1.16), the expansions of Corollary 2.2 take the form*

$$\begin{aligned}\mathcal{G}[\epsilon \zeta_a] \psi_a &= \sum_j g_j \psi_{0j} e_j + \text{c.c.} \\ &\quad + \epsilon \left(\sum_j (g_j \psi_{1j} - (ig_j' \cdot \nabla' + (g_j^2 - |\xi_j|^2) \zeta_{00}) \psi_{0j}) e_j + \sum_{ji} (g_{ji} \psi_{1ji} + A_{ji}) e_{ji} + \text{c.c.} \right)\end{aligned}$$

$$\begin{aligned}
& + \epsilon^2 \left(\sum_j (g_j \psi_{2j} - i g'_j \cdot \nabla' \psi_{1j} - P_j) e_j + \sum_{ji} (g_{ji} \psi_{2ji} - i g'_{ji} \cdot \nabla' \psi_{1ji} + C_{ji}) e_{ji} \right. \\
& \quad \left. + \sum_{jik} (g_{jik} \psi_{2jik} + C_{jik}) e_{jik} + \text{c.c.} - \sqrt{\mu} \Delta' \psi_{00} + C_0 \right) + \epsilon^3 R_2^1
\end{aligned}$$

and

$$\begin{aligned}
& - \mathcal{N}_{\epsilon, \sigma}^2(U_2^a) = \sum_j (-b_j \zeta_{0j}) e_j + \text{c.c.} - \zeta_{00} \\
& + \epsilon \left(\sum_j (-b_j \zeta_{1j} + \frac{1}{B_0} 2i \xi_j \cdot \nabla' \zeta_{0j}) e_j + \sum_{ji} (-b_{ji} \zeta_{1ji} + B_{ji}) e_{ji} + \text{c.c.} - \zeta_{10} + B_0 \right) \\
& + \epsilon^2 \left(\sum_j (-b_j \zeta_{2j} + \frac{2i}{B_0} \xi_j \cdot \nabla' \zeta_{1j} - Q_j) e_j + \sum_{ji} (-b_{ji} \zeta_{2ji} + \frac{2i}{B_0} \xi_{ji} \cdot \nabla' \zeta_{1ji} + D_{ji}) e_{ji} \right. \\
& \quad \left. + \sum_{jik} (-b_{jik} \zeta_{2jik} + D_{jik}) e_{jik} + \text{c.c.} - \zeta_{20} + D_0 \right) + \epsilon^3 R_2^2.
\end{aligned}$$

Here, R_2^1, R_2^2 are the residual terms of Corollary 2.2,

$$\begin{aligned}
A_{jj} &= -(g_{jj} g_j - 2|\xi_j|^2) \zeta_{0j} \psi_{0j}, \quad j \in J, \\
A_{ji} &= -(g_{ji} g_i - \xi_{ji} \cdot \xi_i) \zeta_{0j} \psi_{0i} - (g_{ji} g_j - \xi_{ji} \cdot \xi_j) \zeta_{0i} \psi_{0j}, \quad (j, i) \in I_<, \\
A_{j, -i} &= -(g_{j, -i} g_i + \xi_{j, -i} \cdot \xi_i) \zeta_{0j} \overline{\psi_{0i}} - (g_{j, -i} g_j - \xi_{j, -i} \cdot \xi_j) \overline{\zeta_{0i}} \psi_{0j}, \quad (j, i) \in I_<,
\end{aligned}$$

$$\begin{aligned}
B_0 &= \sum_j (g_j^2 - |\xi_j|^2) |\psi_{0j}|^2, \quad B_{jj} = \frac{1}{2} (g_j^2 + |\xi_j|^2) \psi_{0j}^2, \quad j \in J, \\
B_{ji} &= (g_j g_i + \xi_j \cdot \xi_i) \psi_{0j} \psi_{0i} \quad B_{j, -i} = (g_j g_i - \xi_j \cdot \xi_i) \psi_{0j} \overline{\psi_{0i}}, \quad (j, i) \in I_<
\end{aligned}$$

and

$$\begin{aligned}
P_j &= (H_j + (g_j^2 - |\xi_j|^2) \zeta_{10}) \psi_{0j} + i \xi_j \zeta_{0j} \cdot \nabla' \psi_{00} - C_j, \\
Q_j &= -\frac{1}{B_0} \Delta' \zeta_{0j} + i \xi_j \psi_{0j} \cdot \nabla' \psi_{00} - D_j.
\end{aligned}$$

The exact formulas for the functions C, D are given in the Appendix. In particular, C_j, D_j consist of cubic products of ζ_{0j}, ψ_{0j} and quadratic products of ζ_{1ji}, ψ_{1ji} with ζ_{0k}, ψ_{0k} .

From this proposition and the expansions of the time-derivatives of U_a as exemplified in (2.1), we obtain finally our full expansion of the water waves equation (1.1) with respect to ϵ .

Corollary 2.5. *With Notation 1.1 and that of Proposition 2.4, and with (2.1), the water waves equation (1.1) with the ansatz (1.16) takes the form*

$$\begin{aligned}
& \partial_t \zeta_a - \mathcal{G}[\epsilon \zeta_a] \psi_a = \sum_j (-i \omega_j \zeta_{0j} - g_j \psi_{0j}) e_j + \text{c.c.} \\
& + \epsilon \left(\sum_j (\partial_t' \zeta_{0j} - i \omega_j \zeta_{1j} - g_j \psi_{1j} + (i g'_j \cdot \nabla' + (g_j^2 - |\xi_j|^2) \zeta_{00}) \psi_{0j}) e_j \right. \\
& \quad \left. + \sum_{ji} (-i \omega_{ji} \zeta_{1ji} - g_{ji} \psi_{1ji} - A_{ji}) e_{ji} + \text{c.c.} + \partial_t' \zeta_{00} \right)
\end{aligned}$$

$$\begin{aligned}
& + \epsilon^2 \left(\sum_j (\partial'_t \zeta_{1j} - i\omega_j \zeta_{2j} - g_j \psi_{2j} + i g'_j \cdot \nabla' \psi_{1j} + P_j) e_j \right. \\
& \quad + \sum_{ji} (\partial'_t \zeta_{1ji} - i\omega_{ji} \zeta_{2ji} - g_{ji} \psi_{2ji} + i g'_{ji} \cdot \nabla' \psi_{1ji} - C_{ji}) e_{ji} \\
& \quad + \sum_{jik} (-i\omega_{jik} \zeta_{2jik} - g_{jik} \psi_{2jik} - C_{jik}) e_{jik} + \text{c.c.} + \partial'_t \zeta_{10} + \sqrt{\mu} \Delta' \psi_{00} - C_0 \Big) \\
& + \epsilon^3 r_2^1
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \psi_a + \mathcal{N}_{\epsilon, \sigma}^2(U_2^a) &= \sum_j (-i\omega_j \psi_{0j} + b_j \zeta_{0j}) e_j + \text{c.c.} + \zeta_{00} \\
& + \epsilon \left(\sum_j (\partial'_t \psi_{0j} - i\omega_j \psi_{1j} + b_j \zeta_{1j} - \frac{1}{B_0} 2i\xi_j \cdot \nabla' \zeta_{0j}) e_j \right. \\
& \quad + \sum_{ji} (-i\omega_{ji} \psi_{1ji} + b_{ji} \zeta_{1ji} - B_{ji}) e_{ji} + \text{c.c.} + \partial'_t \psi_{00} + \zeta_{10} - B_0 \Big) \\
& + \epsilon^2 \left(\sum_j (\partial'_t \psi_{1j} - i\omega_j \psi_{2j} + b_j \zeta_{2j} - \frac{1}{B_0} 2i\xi_j \cdot \nabla' \zeta_{1j} + Q_j) e_j \right. \\
& \quad + \sum_{ji} (\partial'_t \psi_{1ji} - i\omega_{ji} \psi_{2ji} + b_{ji} \zeta_{2ji} - \frac{1}{B_0} 2i\xi_{ji} \cdot \nabla' \zeta_{1ji} - D_{ji}) e_{ji} \\
& \quad + \sum_{jik} (-i\omega_{jik} \psi_{2jik} + b_{jik} \zeta_{2jik} - D_{jik}) e_{jik} + \text{c.c.} + \partial'_t \psi_{10} + \zeta_{20} - D_0 \Big) \\
& + \epsilon^3 r_2^2
\end{aligned}$$

with

$$\begin{aligned}
r_2^1 &= \sum_j \partial'_t \zeta_{2j} e_j + \sum_{ji} \partial'_t \zeta_{2ji} e_{ji} + \sum_{jik} \partial'_t \zeta_{2jik} e_{jik} + \text{c.c.} + \partial'_t \zeta_{20} - R_2^1, \\
r_2^2 &= \sum_j \partial'_t \psi_{2j} e_j + \sum_{ji} \partial'_t \psi_{2ji} e_{ji} + \sum_{jik} \partial'_t \psi_{2jik} e_{jik} + \text{c.c.} + \partial'_t \psi_{20} - R_2^2.
\end{aligned}$$

3. FORMAL DERIVATION OF THE MODULATION EQUATIONS

Having determined the expansion in terms of ϵ of the left-hand side of (1.21), we see that the proposed approximation (1.16) satisfies formally the water waves problem (1.1) up to residual terms of order ϵ^3 , i.e., (1.21) holds true, if and only if all terms of the expansion of orders up to ϵ^2 vanish identically. This means that the macroscopic coefficients of each one of the mutually different harmonics have to vanish separately. According to Notation 1.1(3a), which stipulates that all plane-waves e_j , $j \in J$, are mutually different and also different from the zeroth-order harmonic $e^0 = 1$, and our closedness assumption (Notation 1.1(3c)) that all higher-order harmonics are neither plane-waves nor equal to 1, this implies that each of the coefficients to e_j , $1 = e^0$ and the different higher harmonics of orders up to ϵ^2 has to vanish identically. This necessary condition, leads to equations for the corresponding macroscopic coefficients, which are called modulation equations.

The typical way to obtain these equations is to proceed step by step from lower to higher orders of ϵ^n , $n = 0, 1, 2$, and require at each step that the coefficients to

each different harmonic vanishes. However, one could as well add for each harmonic some of the macroscopic coefficients multiplied by their orders ϵ^n and require that the sum vanishes up to residuals of a higher order. Since at each step the obtained macroscopic equations are typically undetermined, in the sense that they contain terms which are determined at a higher order of ϵ^n , the second approach allows for a different choice of the, at order ϵ^2 , still undetermined coefficients. However, with both approaches we obtain (1.21). For more details on the second approach we refer to Remark 3.3. In the present paper, we follow the first, more standard approach.

Starting from the terms of order ϵ^0 , and according to our assumptions, we obtain immediately from Corollary 2.5 the macroscopic equations

$$(3.1) \quad \zeta_{0j} = i \frac{\omega_j}{b_j} \psi_{0j}, \quad \omega_j^2 = b_j g_j \quad \text{for } j \in J, \quad \text{and} \quad \zeta_{00} = 0.$$

At the next order ϵ^1 , by requiring that the coefficients of e_j vanish, and using (3.1) and the identity (1.17), we obtain by elimination of

$$b_j \zeta_{1j} - i \omega_j \psi_{1j} = \frac{b_j}{i \omega_j} (i \omega_j \zeta_{1j} + g_j \psi_{1j})$$

(due to $\omega_j^2 = b_j g_j$) the equations

$$(3.2) \quad \partial_t' \psi_{0j} + \nabla \omega_j \cdot \nabla' \psi_{0j} = 0,$$

$$(3.3) \quad \zeta_{1j} = i \frac{\omega_j}{b_j} \psi_{1j} + \nabla_{B_0} \omega_j \cdot \nabla' \psi_{0j}$$

(with the notation (1.19)). For the coefficient of $e^0 = 1$ we get

$$(3.4) \quad \partial_t' \psi_{00} + \zeta_{10} = B_0 = \sum_j (g_j^2 - |\xi_j|^2) |\psi_{0j}|^2$$

(see Proposition 2.4 for the definition of B_0). Moreover, for the coefficients of the second-order harmonics e_{ji} , $(j, i) \in I$, which surely differ from the first- and zero-order harmonics by the first non-resonance condition of Notation 1.1(3c), we obtain the equations

$$(3.5) \quad \begin{pmatrix} \zeta_{1ji} \\ \psi_{1ji} \end{pmatrix} = \frac{1}{\omega_{ji}^2 - b_{ji} g_{ji}} \begin{pmatrix} i \omega_{ji} & -g_{ji} \\ b_{ji} & i \omega_{ji} \end{pmatrix} \begin{pmatrix} A_{ji} \\ B_{ji} \end{pmatrix}$$

with A_{ji}, B_{ji} as in Proposition 2.4. In particular, due to (3.1), we have

$$(3.6) \quad \begin{aligned} A_{jj} &= -i(g_{jj} g_j - 2|\xi_j|^2) \frac{\omega_j}{b_j} \psi_{0j}^2, \quad j \in J, \\ A_{ji} &= -i \left((g_{ji} g_i - \xi_{ji} \cdot \xi_i) \frac{\omega_j}{b_j} + (g_{ji} g_j - \xi_{ji} \cdot \xi_j) \frac{\omega_i}{b_i} \right) \psi_{0j} \psi_{0i}, \quad (j, i) \in I_{<}, \\ A_{j,-i} &= -i \left((g_{j,-i} g_i + \xi_{j,-i} \cdot \xi_i) \frac{\omega_j}{b_j} - (g_{j,-i} g_j - \xi_{j,-i} \cdot \xi_j) \frac{\omega_i}{b_i} \right) \psi_{0j} \overline{\psi_{0i}}, \quad (j, i) \in I_{<}. \end{aligned}$$

REMARK 3.1. Note, that we do not require that all second-order harmonics are mutually different, since if two or more coincide, corresponding to a subset $\Lambda \subset I$ of indices, and are different from all the other, then we consider in (1.16) only one

representative $(\ell, m) \in \Lambda$ and replace on the right-hand side of (3.5) $(A_{\ell m}, B_{\ell m})^T$ by $\sum_{(\lambda, \mu) \in \Lambda} (A_{\lambda \mu}, B_{\lambda \mu})^T$. \square

Commenting on the results obtained at the order ϵ^1 , we note that (3.2) describes only the macroscopic transport of the leading- (ϵ^0) -order amplitudes of the plane-waves e_j , $j \in J$, with the group velocity of the wave $\nabla \omega_j$. Hence, at leading order the macroscopic amplitudes do not interact. Of course, this is due to our non-resonance condition, viz. the first inequality in Notation 1.1(3c).

Quadratic interaction of the plane-wave-amplitudes can be detected only in the zeroth- and second-order harmonics of order ϵ via (3.4), (3.5). Moreover, we see that (3.4) contains also the time derivative of the leading-order non-oscillating part of the velocity potential at the surface. However, we realize that this equation is not yet closed at the level ϵ . The same applies for (3.3), which relates the ϵ -order corrections of the amplitudes to their leading order transport term.

REMARK 3.2. Up to now the ϵ^2 -terms of U_a in (1.16) did not contribute to the expansion of the water waves problem as given in Corollary 2.5. The same holds true for ψ_{10} . Hence, if we are interested only in the leading-order modulation equations we can consider the approximation $U_{a,1}$, defined as U_a but without ϵ^2 -terms, and stop the derivation procedure here, choosing to set $\psi_{00} = \psi_{1j} = \psi_{10} = 0$. Thus, we have obtained up to now that the approximation

$$U_{a,1} = \sum_j \begin{pmatrix} i\frac{\omega_j}{b_j} \\ 1 \end{pmatrix} \psi_{0j} e_j + \text{c.c.} + \epsilon \left(\sum_j \begin{pmatrix} \zeta_{1j} \\ 0 \end{pmatrix} e_j + \sum_{ji} \begin{pmatrix} \zeta_{1ji} \\ \psi_{1ji} \end{pmatrix} e_{ji} + \text{c.c.} + \begin{pmatrix} \zeta_{10} \\ 0 \end{pmatrix} \right)$$

with the appearing functions determined by (3.2)–(3.5) (with $\psi_{1j} = \psi_{00} = 0$) satisfies the water waves problem (1.1) up to residual terms of order ϵ^2 , i.e.

$$\partial_t U_{a,1} + \mathcal{N}_\sigma(U_{a,1}) = \epsilon^2 (r_1^1, r_1^2)^T$$

with

$$\begin{aligned} r_1^1 &= \sum_j \partial_t' \zeta_{1j} e_j + \sum_{ji} \partial_t' \zeta_{1ji} e_{ji} + \text{c.c.} + \partial_t' \zeta_{10} - R_1^1, \\ r_1^2 &= \sum_{ji} \partial_t' \psi_{1ji} e_{ji} + \text{c.c.} - R_1^2, \end{aligned}$$

and where R_1^1, R_1^2 are defined by (2.11), (2.14) in the proof of Corollary 2.2. \square

According to the comments made above, we expect that the interaction of three non-resonant modulated waves appears macroscopically at the first-order corrections to the leading-order amplitudes. Remark 3.2. implies that in order to obtain the corresponding modulation equations, we have to carry our derivation procedure to the next order, i.e. we need to eliminate also the ϵ^2 -terms of the expansions given in Corollary 2.5. Due to our assumptions in Notation 1.1(3a,c), this has to be done separately for each e_j , $j \in J$, the zeroth-order harmonic and the higher-order harmonics.

Equating the coefficients of the first-order harmonics e_j of order ϵ^2 to zero, and using the equations (3.1), (3.2), (3.3), (3.4) and the identities (1.17), (1.18) with the notation (1.19), we obtain by elimination of $b_j \zeta_{2j} - i\omega_j \psi_{2j}$, that the two equations

can be written equivalently in the form

$$(3.7) \quad \partial'_t \psi_{1j} + \nabla \omega_j \cdot \nabla' \psi_{1j} = E_j,$$

$$(3.8) \quad \zeta_{2j} = i \frac{\omega_j}{b_j} \psi_{2j} + \nabla_{\text{Bo}} \omega_j \cdot \nabla' \psi_{1j} + F_j,$$

with

$$(3.9) \quad E_j = i \frac{1}{2} \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} - i \psi_{0j} \left(\frac{b_j}{2\omega_j} (g_j^2 - |\xi_j|^2) \partial'_t + \xi_j \cdot \nabla' \right) \psi_{00} + \tilde{E}_j$$

where

$$(3.10) \quad \tilde{E}_j = i \frac{b_j}{2\omega_j} (g_j^2 - |\xi_j|^2) B_0 \psi_{0j} - i \frac{b_j}{2\omega_j} C_j + \frac{1}{2} D_j,$$

and

$$\begin{aligned} F_j = & -i \frac{1}{2b_j} \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} + i \frac{1}{b_j} \frac{1}{\text{Bo}} \left(2\xi_j \cdot \nabla' (\nabla_{\text{Bo}} \omega_j \cdot \nabla' \psi_{0j}) + \frac{\omega_j}{b_j} \Delta' \psi_{0j} \right) \\ & + i \frac{1}{2\omega_j} (g_j^2 - |\xi_j|^2) \psi_{0j} \partial'_t \psi_{00} - i \frac{1}{2\omega_j} (g_j^2 - |\xi_j|^2) B_0 \psi_{0j} + i \frac{1}{2\omega_j} C_j + \frac{1}{2b_j} D_j. \end{aligned}$$

We note that the solution of (3.7) closes now equation (3.3) and determines ζ_{1j} . In particular, \tilde{E}_j and the three last terms of F_j consist of cubic products of ψ_{0j} (cf. Proposition 2.4, (3.1), (3.5) and (3.6)). Hence, E_j, F_j depend only on ψ_{0j} and ψ_{00} .

While ψ_{0j} is determined by (3.2), we obtain from the coefficient of the e^0 -term of order ϵ^2 in the first expansion of Corollary 2.5 and (3.4), by elimination of $\partial'_t \zeta_{10}$, the second-order inhomogeneous wave equation

$$(3.11) \quad \partial_t'^2 \psi_{00} - \sqrt{\mu} \Delta' \psi_{00} = \partial'_t B_0 - C_0 = \sum_j \left((g_j^2 - |\xi_j|^2) \partial'_t + 2 \frac{\omega_j}{b_j} \xi_j \cdot \nabla' \right) |\psi_{0j}|^2.$$

Its solution determines then ζ_{10} through (3.4). Moreover, for the coefficient of e^0 at order ϵ^2 in the second expansion of Corollary 2.5 we obtain

$$(3.12) \quad \partial'_t \psi_{10} + \zeta_{20} = D_0 = \sum_j (i(\xi_j - g_j g'_j) \cdot \nabla' \psi_{0j} + (g_j^2 - |\xi_j|^2) \psi_{1j}) \overline{\psi_{0j}} + \text{c.c.}$$

Finally, equating the coefficients of the higher harmonics to zero, we obtain

$$(3.13) \quad \begin{aligned} \begin{pmatrix} \zeta_{2ji} \\ \psi_{2ji} \end{pmatrix} &= \frac{1}{\omega_{ji}^2 - b_{ji} g_{ji}} \begin{pmatrix} i\omega_{ji} & -g_{ji} \\ b_{ji} & i\omega_{ji} \end{pmatrix} \begin{pmatrix} C_{ji} - \partial'_t \zeta_{1ji} - i g'_{ji} \cdot \nabla' \psi_{1ji} \\ D_{ji} - \partial'_t \psi_{1ji} + \frac{1}{\text{Bo}} 2i \xi_{ji} \cdot \nabla' \zeta_{1ji} \end{pmatrix}, \\ \begin{pmatrix} \zeta_{2jik} \\ \psi_{2jik} \end{pmatrix} &= \frac{1}{\omega_{jik}^2 - b_{jik} g_{jik}} \begin{pmatrix} i\omega_{jik} & -g_{jik} \\ b_{jik} & i\omega_{jik} \end{pmatrix} \begin{pmatrix} C_{jik} \\ D_{jik} \end{pmatrix}, \end{aligned}$$

where the right-hand sides depend only on the functions ψ_{0j} (and its first order derivatives) and ψ_{1j} , see Appendix and (3.3), (3.5). The case where two or more higher harmonics coincide is treated as in Remark 3.1. In the case where only the first non-resonance condition of Notation 1.1(3c) holds true, i.e. when some third-order harmonic e_{jik} equals e_j , the corresponding C_{jik}, D_{jik} of the expansions in Corollary 2.5 contribute to (3.7), (3.8) in the same way as the C_j, D_j , and the $\zeta_{2jik}, \psi_{2jik}$ can be set equal to zero. According to the Appendix, the terms C_{jik}, D_{jik} are cubic products of the leading-order amplitudes $\psi_{0j}, j \in J$.

In analogy to the procedure in Remark 3.2. at the level ϵ , we can close the system of modulation equations derived so far, by setting $\psi_{10} = \psi_{2j} = \psi_{20} = 0$. Thus, we have obtained that the second order approximation

$$(3.14) \quad \begin{aligned} U_a = & \sum_j \begin{pmatrix} i\frac{\omega_j}{b_j} \\ \psi_{0j} \\ 1 \end{pmatrix} \psi_{0j} e_j + \text{c.c.} + \begin{pmatrix} 0 \\ \psi_{00} \end{pmatrix} \\ & + \epsilon \left(\sum_j \begin{pmatrix} \zeta_{1j} \\ \psi_{1j} \end{pmatrix} e_j + \sum_{ji} \begin{pmatrix} \zeta_{1ji} \\ \psi_{1ji} \end{pmatrix} e_{ji} + \text{c.c.} + \begin{pmatrix} \zeta_{10} \\ 0 \end{pmatrix} \right) \\ & + \epsilon^2 \left(\sum_j \begin{pmatrix} \zeta_{2j} \\ 0 \end{pmatrix} e_j + \sum_{ji} \begin{pmatrix} \zeta_{2ji} \\ \psi_{2ji} \end{pmatrix} e_{ji} + \sum_{jik} \begin{pmatrix} \zeta_{2jik} \\ \psi_{2jik} \end{pmatrix} e_{jik} + \text{c.c.} + \begin{pmatrix} \zeta_{20} \\ 0 \end{pmatrix} \right) \end{aligned}$$

with the appearing macroscopic functions determined as above (with $\psi_{2j} = 0$ in (3.8) and $\psi_{10} = 0$ in (3.12)) is consistent with the water waves problem (1.1), in the sense that it satisfies

$$(3.15) \quad \partial_t U_a + \mathcal{N}_{\epsilon, \sigma}(U_a) = \epsilon^3 (r_2^1, r_2^2)^T,$$

where

$$(3.16) \quad \begin{aligned} r_2^1 &= \sum_j \partial_t' \zeta_{2j} e_j + \sum_{ji} \partial_t' \zeta_{2ji} e_{ji} + \sum_{jik} \partial_t' \zeta_{2jik} e_{jik} + \text{c.c.} + \partial_t' \zeta_{20} - R_2^1, \\ r_2^2 &= \sum_{ji} \partial_t' \psi_{2ji} e_{ji} + \sum_{jik} \partial_t' \psi_{2jik} e_{jik} + \text{c.c.} - R_2^2 \end{aligned}$$

with R_2^1, R_2^2 as in Corollary 2.2.

In particular, all macroscopic functions can be determined if we solve the system

$$(3.17) \quad \begin{cases} \partial_t' \psi_{0j} + \nabla \omega_j \cdot \nabla' \psi_{0j} = 0, \\ \partial_t'^2 \psi_{00} - \sqrt{\mu} \Delta' \psi_{00} = \sum_j \left((g_j^2 - |\xi_j|^2) \partial_t' + 2 \frac{\omega_j}{b_j} \xi_j \cdot \nabla' \right) |\psi_{0j}|^2, \\ \partial_t' \psi_{1j} + \nabla \omega_j \cdot \nabla' \psi_{1j} = E_j \end{cases}$$

with $j \in J = \{1, 2, 3\}$ and

$$(3.9) \quad E_j = i \frac{1}{2} \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} - i \psi_{0j} \left(\frac{b_j}{2\omega_j} (g_j^2 - |\xi_j|^2) \partial_t' + \xi_j \cdot \nabla' \right) \psi_{00} + \tilde{E}_j$$

with \tilde{E}_j as in (3.10).

We close this section with the remark mentioned at the beginning of the section, concerning an alternative derivation procedure in comparison to the one presented here.

REMARK 3.3. In the approach presented above, we derived *consecutively* the macroscopic equations that make the first-order-harmonic terms vanish *at each* order ϵ^1 and ϵ^2 *separately*. In a different approach, which in the case of a single carrier wave leads to the so-called *Benney-Roskes system* ([6], cf. also [37, §8.2.5] and the references given therein), one considers *jointly* the coefficients of the first-order-harmonics of orders ϵ^1 and ϵ^2 and derives conditions such that these joint coefficients vanish up to terms of order $O(\epsilon^3)$. More precisely, starting from the

coefficients of the first-order harmonics of orders ϵ^1 , ϵ^2 of Corollary 2.5, with (3.1), (1.17), (1.19) and (1.18) one can write equivalently

$$\begin{aligned} & (\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} - \epsilon i \frac{1}{2\omega_j} (\partial'_t + \nabla \omega_j \cdot \nabla') (b_j \zeta_{1j} + i\omega_j \psi_{1j}) \\ & + \epsilon i \frac{b_j}{2\omega_j} \nabla_{\text{Bo}} \omega_j \cdot \nabla' (b_j \zeta_{1j} - i\omega_j \psi_{1j} - b_j \nabla_{\text{Bo}} \omega_j \cdot \nabla' \psi_{0j}) - \epsilon E_j = O(\epsilon^2), \end{aligned}$$

$$\begin{aligned} & (\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} + b_j \zeta_{1j} - i\omega_j \psi_{1j} - b_j \nabla_{\text{Bo}} \omega_j \cdot \nabla' \psi_{0j} \\ & + \epsilon (b_j \zeta_{2j} - i\omega_j \psi_{2j}) + \epsilon (\partial'_t \psi_{1j} - \frac{1}{\text{Bo}} 2i\xi_j \cdot \nabla' \zeta_{1j} + Q_j) = O(\epsilon^2) \end{aligned}$$

with E_j as in (3.9) (although typically with the second and fourth term written together as $i \frac{b_j}{2\omega_j} (g_j^2 - |\xi_j|^2) \zeta_{10} \psi_{0j}$, see (3.4)).

At leading order in ϵ we obtain from the two equations (consecutively)

$$\partial'_t \psi_{0j} + \nabla \omega_j \cdot \nabla' \psi_{0j} = O(\epsilon), \quad b_j \zeta_{1j} - i\omega_j \psi_{1j} - b_j \nabla_{\text{Bo}} \omega_j \cdot \nabla' \psi_{0j} = O(\epsilon)$$

(in analogy to (3.2), (3.3)). Thus, requiring that the equation on the right is satisfied *exactly* (i.e., without $O(\epsilon)$ -terms), we obtain (3.3) and

$$(\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} - \epsilon i \frac{1}{2\omega_j} (\partial'_t + \nabla \omega_j \cdot \nabla') (b_j \zeta_{1j} + i\omega_j \psi_{1j}) = \epsilon E_j + O(\epsilon^2),$$

$$b_j \zeta_{2j} - i\omega_j \psi_{2j} = -\frac{1}{\epsilon} (\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} - \partial'_t \psi_{1j} + \frac{1}{\text{Bo}} 2i\xi_j \cdot \nabla' \zeta_{1j} - Q_j + O(\epsilon).$$

The second equation plays the same rôle as (3.8). Choosing $b_j \zeta_{2j} + i\omega_j \psi_{2j}$ arbitrarily, we can determine ζ_{2j} and ψ_{2j} .

Now, we require

$$(3.18) \quad (\partial'_t + \nabla \omega_j \cdot \nabla') (b_j \zeta_{1j} + i\omega_j \psi_{1j}) = 0.$$

Together with (3.3), this determines ζ_{1j} and ψ_{1j} (in difference to our approach above, using (3.3), (3.7) for that), and yields

$$(3.19) \quad (\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} = \epsilon E_j + O(\epsilon^2).$$

According to Lannes, see [37, §8.2.3., fn. 9] and [35], the choice (3.18) is the only possible in order to avoid the secular growth of $\epsilon(b_j \zeta_{1j} + i\omega_j \psi_{1j})$ on time scales of order $t'' = \epsilon t' = \epsilon^2 t$. This implies that the results obtained in the present paper are relevant only on time scales of order $t' = \epsilon t$. Note, that with (3.18) we obtain from (3.3) and (3.19)

$$(\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{1j} = \epsilon i \frac{b_j}{2\omega_j} \nabla_{\text{Bo}} \omega_j \cdot \nabla' E_j + \mathcal{O}(\epsilon^2)$$

instead of (3.7) with our approach. However, in both cases we obtain

$$(\partial'_t + \nabla \omega_j \cdot \nabla') (\psi_{0j} + \epsilon \psi_{1j}) = \epsilon E_j + \mathcal{O}(\epsilon^2).$$

The system

$$\begin{cases} (\partial'_t + \nabla \omega_j \cdot \nabla') \psi_{0j} = \epsilon E_j, \\ \partial'_t \zeta_{10} + \sqrt{\mu} \Delta' \psi_{00} = -2 \sum_j \frac{\omega_j}{b_j} \xi_j \cdot \nabla' |\psi_{0j}|^2, \\ \partial'_t \psi_{00} + \zeta_{10} = \sum_j (g_j^2 - |\xi_j|^2) |\psi_{0j}|^2 \end{cases}$$

with E_j as in (3.9) (in the form mentioned above) can be seen as the *Benney-Roskes system for three capillary-gravity water waves*. The original Benney-Roskes system

was derived for a single gravity water wave (see [6] and [37, (8.34)]), and can be recovered from the system above by setting the surface tension to zero and assuming that two of the three waves are identically vanishing. For its well-posedness we refer to [43]. Note, that in contrast to the system (3.17) that we consider in this paper, the equations of the Benney-Roskes system are coupled. \square

4. JUSTIFICATION IN THE CASE WITHOUT SURFACE TENSION

For the justification of the modulation equations derived in Section 3 in the case of pure gravity waves (i.e. without surface tension, $\sigma = \frac{1}{\text{Bo}} = 0$) we use the well-posedness result for gravity water waves of finite depth obtained by Lannes and Alvarez-Samaniego in [36, 2, 3, 37] as presented in [37, Theorems 4.16, 4.18]. There, the well-posedness is established with respect to the energy norm

$$(4.1) \quad \mathcal{E}_\epsilon^N(U) = |\mathfrak{P}\psi|_{H^{t_0+3/2}}^2 + \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} |\partial^\alpha \zeta|_2^2 + |\mathfrak{P}\psi_{\epsilon,(\alpha)}|_2^2, \quad N \in \mathbb{N},$$

with $U = (\zeta, \psi)^T$,

$$(4.2) \quad \begin{aligned} \psi_{(0)} &= \psi, & \psi_{\epsilon,(\alpha)} &= \partial^\alpha \psi - (w[\epsilon\zeta]\epsilon\psi)\partial^\alpha \zeta \quad \text{for } \alpha \neq 0, \\ w[\zeta]\psi &= \frac{\mathcal{G}[\zeta]\psi + \nabla\zeta \cdot \nabla\psi}{1 + |\nabla\zeta|^2} \quad \text{and} \quad \mathfrak{P} = \frac{|D|}{(1 + \sqrt{\mu}|D|)^{1/2}} \end{aligned}$$

(recall the Fourier-multiplier notation (1.10)).

$H^s(\mathbb{R}^d)$ ($s \in \mathbb{R}$) are the fractional Sobolev spaces (see, e.g., [31])

$$(4.3) \quad H^s(\mathbb{R}^d) = \{u \in \mathfrak{S}'(\mathbb{R}^d) : |u|_{H^s} = |\Lambda^s u|_2 < \infty\}, \quad |u|_2^2 = \int_{\mathbb{R}^d} |u(X)|^2 dX,$$

where $\mathfrak{S}'(\mathbb{R}^d)$ is the space of tempered distributions and $\Lambda = \langle D \rangle = (1 + |D|^2)^{1/2}$ is the fractional derivative.

Associated to the energy norm (4.1) is the space of time-dependent functions that remain bounded with respect to this norm up to the time $T > 0$

$$E_{\epsilon,T}^N = \{U \in C([0, T]; H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d)) : \mathcal{E}_\epsilon^N(U(\cdot)) \in L^\infty([0, T])\}$$

and the space of initial data

$$E_{\epsilon,0}^N = \{U^0 \in H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d) : \mathcal{E}_\epsilon^N(U^0) < \infty\}.$$

Here, $\dot{H}^{s+1}(\mathbb{R}^d)$ ($s \in \mathbb{R}$) are the Beppo-Levi topological vector spaces

$$\dot{H}^{s+1}(\mathbb{R}^d) = \{f \in L_{loc}^2(\mathbb{R}^d) : \nabla f \in H^s(\mathbb{R}^d)^d\}$$

endowed with the semi-norm $|f|_{\dot{H}^{s+1}} = |\nabla f|_{H^s}$, which implies that $\dot{H}^{s+1}(\mathbb{R}^d)/\mathbb{R}$ are Banach spaces (see, e.g., [37, §2.1.2.] and [17]).

The well-posedness result of [37, Th. 4.16, 4.18] can be adapted to the case of deep water ($\mu \geq 1$) of finite depth (with flat bottom, and full transversality of the waves if $d = 2$) as follows (see also Remark 4.3. below).

Theorem 4.1. *Let $t_0 > d/2$, $t_0 \geq 1$, $N \geq t_0 + t_0 \vee 2 + 3/2$, $1 \leq \mu \leq \mu_{\max} < \infty$, and $0 < \epsilon \leq 1$. Assume that $U^0 = (\zeta^0, \psi^0)^T \in E_{\epsilon,0}^N$ with*

$$(4.4) \quad 1 - \epsilon|\zeta^0|_\infty \geq h_{\min} > 0 \quad \text{and} \quad \mathfrak{a}_\epsilon(U^0) \geq a_0 > 0,$$

where $\mathbf{a}_\epsilon(U^0)$ is defined in Remark 4.2 below. Then, there exists $T > 0$ and a unique solution $U = (\zeta, \psi)^T \in E_{\epsilon, T/\epsilon}^N$ to the water waves problem

$$(4.5) \quad \partial_t U + \mathcal{N}_{\epsilon, 0}(U) = 0, \quad \mathcal{N}_{\epsilon, 0}(U) = \begin{pmatrix} -\mathcal{G}[\epsilon\zeta]\psi \\ \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2} \frac{(\mathcal{G}[\epsilon\zeta]\psi + \epsilon \nabla\zeta \cdot \nabla\psi)^2}{1 + \epsilon^2 |\nabla\zeta|^2} \end{pmatrix}$$

with initial data U^0 , and

$$\frac{1}{T} = c_1, \quad \sup_{t \in [0, T/\epsilon]} \mathcal{E}_\epsilon^N(U(t)) = c_2,$$

where the constants $c_j = C(\mathcal{E}_\epsilon^N(U^0), \mu_{\max}, h_{\min}^{-1}, a_0^{-1})$, $j = 1, 2$, are non-decreasing functions of their arguments.

Furthermore, if there exists $U_{app} = (\zeta_{app}, \psi_{app})^T \in E_{\epsilon, T/\epsilon}^N$ such that

$$1 - \epsilon \sup_{t \in [0, T/\epsilon]} |\zeta_{app}(t)|_\infty > 0 \quad \text{and} \quad \partial_t U_{app} + \mathcal{N}_{\epsilon, 0}(U_{app}) = (r^1, r^2)^T$$

with $(r^1, \mathfrak{P}r^2) \in L^\infty([0, T/\epsilon]; H^N(\mathbb{R}^d)^2)$, and a constant $c_{app} > 0$ such that

$$\sup_{t \in [0, T/\epsilon]} \mathcal{E}_\epsilon^N(U_{app}(t)) \leq c_{app},$$

then the error $\mathbf{e} = U - U_{app}$ satisfies for all $t \in [0, T/\epsilon]$

$$\mathcal{E}_\epsilon^{N-1}(\mathbf{e}(t))^{1/2} \leq C(c_2, c_{app}) \left(\mathcal{E}_\epsilon^{N-1}(\mathbf{e}(0))^{1/2} + t |(r^1, \mathfrak{P}r^2)|_{L^\infty([0, t]; H^N)} \right)$$

with C a non-decreasing function of its arguments.

REMARK 4.1. We introduced the index ϵ in the notation of the energy norm, the corresponding spaces and $\mathcal{N}_{\epsilon, 0}$, in order to point out their dependence on this parameter. (Our justification result below uses the above theorem with $\epsilon = 1$.) However, it is important to note that the time T and the constants c_j , $j = 1, 2$, in the statement of the theorem are independent of $\epsilon \in (0, 1]$, since $\mathcal{E}_\epsilon^N(U)$ is bounded with respect to ϵ . This is shown easily in the case $U = (\zeta, \psi)^T \in H^{N+1} \times \dot{H}^{N+1}(\mathbb{R}^d)$. Indeed, from (4.2) we obtain

$$(4.6) \quad |\mathfrak{P}u|_{H^s} \leq |\nabla u|_{H^s} \quad \forall s \in \mathbb{R}, \quad \mu \geq 0,$$

and hence, with Lemma 2.3 (1) and (4), and (2.7) for $\mu \geq 1$ and $t_0 \geq 1$

$$|\mathfrak{P}((w[\epsilon\zeta]\psi)\partial^\alpha\zeta)|_2 \leq C(h_{\min}^{-1}, \mu_{\max}, |\epsilon\zeta|_{H^{t_0+1}}) |\nabla\psi|_{H^{t_0}} |\partial^\alpha\zeta|_{H^1}.$$

This yields in particular,

$$\mathcal{E}_\epsilon^N(U) = \mathcal{E}_0^N(U) + O(\epsilon) \quad \text{as } \epsilon \rightarrow 0$$

but also

$$(4.7) \quad \mathcal{E}_\epsilon^N(U) \leq C(h_{\min}^{-1}, \mu_{\max}, |\epsilon U|_{H^{t_0+1} \times \dot{H}^{t_0+1}}) |U|_{H^{N+1} \times \dot{H}^{N+1}}^2$$

for $N \geq t_0 + 3/2$. □

REMARK 4.2. The first condition in (4.4) implies that the water height never vanishes. (Recall that $\zeta^0 \in H^{t_0}(\mathbb{R}^d) \subset C \cap L^\infty(\mathbb{R}^d)$ for $t_0 > d/2$.) For given ζ^0 , it is uniformly satisfied for $0 < \epsilon \leq \epsilon_0$, with $0 < \epsilon_0 \leq 1$, if $1 - \epsilon_0 |\zeta^0|_\infty \geq h_{\min} > 0$. However, note that $h_{\min} > 0$ influences the existence time $T > 0$ of solutions to the water waves problem.

Moreover, the well-posedness of the water-waves problem and the existence time of its solution depend crucially on the validity of the (strict-)hyperbolicity condition

$$(4.8) \quad \mathbf{a}_\epsilon(U^0) = \mathbf{a}_1(\epsilon U^0) = 1 - \mathbf{b}_1(\epsilon U^0) \geq a_0 > 0$$

with

$$\begin{aligned} \mathbf{b}_1(U) &= w[\zeta] \mathcal{N}_{1,0}^2(U) - (\nabla \psi - (w[\zeta] \psi) \nabla \zeta) \cdot \nabla (w[\zeta] \psi) \\ &+ \frac{\mathcal{G}[\zeta]((w[\zeta] \psi) \mathcal{G}[\zeta] \psi) + (\mathcal{G}[\zeta] \psi) \nabla \cdot (\nabla \psi - (w[\zeta] \psi) \nabla \zeta) + (\nabla \zeta \cdot \nabla \mathcal{G}[\zeta] \psi) w[\zeta] \psi}{1 + |\nabla \zeta|^2}, \end{aligned}$$

where $\mathcal{N}_{1,0}^2(U)$ is the second component of $\mathcal{N}_{1,0}(U)$ given in (4.5).

Using the estimates

$$|\mathcal{G}[\zeta] \psi|_{H^s}, |w[\zeta] \psi|_{H^s} \leq C(h_{\min}^{-1}, \mu_{\max}, |\zeta|_{H^{s \vee t_0+1}}) |\nabla \psi|_{H^s}, \quad s \geq 0,$$

which follow from (2.7) and Lemma 2.3 (1) and (4), we obtain

$$|\mathbf{b}_1(U)|_\infty \leq C|\mathbf{b}_1(U)|_{H^{t_0}} \leq C(h_{\min}^{-1}, \mu_{\max}, |\zeta|_{H^{t_0+2}})(|\zeta|_{H^{t_0+1}} + |\nabla \psi|_{H^{t_0+1}}^2),$$

and, hence,

$$\mathbf{a}_\epsilon(U^0) \geq 1 - \epsilon C(h_{\min}^{-1}, \mu_{\max}, |\zeta^0|_{H^{t_0+2}})(|\zeta^0|_{H^{t_0+1}} + \epsilon |\nabla \psi^0|_{H^{t_0+1}}^2) \geq a_0 > 0.$$

Thus, also here, for given U^0 , there exists an $\epsilon_0 \leq 1$, such that the second condition in (4.4) holds true uniformly for $0 \leq \epsilon \leq \epsilon_0$. However, the constant $a_0 > 0$ influences the existence time $T > 0$ of the solution of (4.5). For more information on the definition, the rôle, and the properties of $\mathbf{a}_\epsilon(U)$ we refer the reader to [37, §§ 4.2.3, 4.3.1, 4.3.5]. \square

REMARK 4.3. As mentioned above, Theorem 4.1 is an adaptation of [37, Th. 4.16, 4.18]. There, the main focus are applications to shallow water theory, which corresponds to $\nu \sim 1$ for the parameter $\nu = \tanh(2\pi\sqrt{\mu})/(2\pi\sqrt{\mu})$ in the general nondimensionalized form for the water waves problem set up in [37, 2], viz.

$$\partial_t U + \tilde{\mathcal{N}}_\nu(U) = 0, \quad \tilde{\mathcal{N}}_\nu(U) = \left(\zeta + \frac{\varepsilon}{2\nu} |\nabla \psi|^2 - \frac{\frac{1}{\mu\nu} \mathcal{G}[\varepsilon \zeta] \psi}{2\mu\nu} - \frac{\frac{\varepsilon}{2\mu\nu} (\mathcal{G}[\varepsilon \zeta] \psi + \varepsilon \mu \nabla \zeta \cdot \nabla \psi)^2}{1 + \varepsilon^2 \mu |\nabla \zeta|^2} \right),$$

and hence Theorems 4.16, 4.18 in [37] are formulated for the case $\nu = 1$. However, these results, *as well as their method of proof*, hold true also for *deep water of finite depth*, $\nu \sim (2\pi\sqrt{\mu})^{-1}$ or (after suitable renormalization of the equations, see [37, Ch. 4, fn. 9]), equivalently, $\nu \sim \mu^{-1/2}$, leading to

$$\tilde{\mathcal{N}}_{\frac{1}{\sqrt{\mu}}}(U) = \left(\zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\frac{1}{\sqrt{\mu}} \mathcal{G}[\frac{\varepsilon}{\sqrt{\mu}} \zeta] \psi}{\frac{1}{\sqrt{\mu}} \mathcal{G}[\frac{\varepsilon}{\sqrt{\mu}} \zeta] \psi + \varepsilon \nabla \zeta \cdot \nabla \psi} - \frac{\frac{\varepsilon}{2} (\frac{1}{\sqrt{\mu}} \mathcal{G}[\frac{\varepsilon}{\sqrt{\mu}} \zeta] \psi + \varepsilon \nabla \zeta \cdot \nabla \psi)^2}{1 + \varepsilon^2 |\nabla \zeta|^2} \right) = \mathcal{N}_{\epsilon,0}(U), \quad \epsilon = \varepsilon \sqrt{\mu},$$

provided μ is bounded, since then, $\nu = \tanh(2\pi\sqrt{\mu})/(2\pi\sqrt{\mu})$ remains of order $\mathcal{O}(1)$ with respect to μ (see [37, Remark 8.7]). Nevertheless, an analogous result holds true if $\epsilon = \varepsilon \sqrt{\mu}$ is bounded (even when $\mu \rightarrow \infty$), although the proof for this needs a different approach (see [37, §4.4.3] and [2, 3]). However, since here we are interested in the case of finite depth $\sqrt{\mu} < \infty$, we use the well-posedness result as presented above, and hence in our justification result we can not simply take the limit $\mu \rightarrow \infty$.

Since (4.5) is equivalent to $\partial_t(\epsilon U) + \mathcal{N}_{1,0}(\epsilon U) = 0$ (note the index $\epsilon = 1$ in $\mathcal{N}_{1,0}$) with $\mathcal{E}_1^N(\epsilon U) = \epsilon^2 \mathcal{E}_\epsilon^N(U)$ and $\mathbf{a}_1(\epsilon U) = \mathbf{a}_\epsilon(U)$, we realize that the solution U in

the first part of the theorem corresponds to the unique solution $\epsilon U \in E_{1,T/\epsilon}^N$ of the initial value problem

$$\partial_t(\epsilon U) + \mathcal{N}_{1,0}(\epsilon U) = 0, \quad \epsilon U(0) = \epsilon U^0 \in E_{1,0}^N$$

with T, c_2 as in the theorem, i.e. independent of $\epsilon \in (0, 1]$.

Moreover, from (3.15) we obtain

$$\partial_t(\epsilon U_a) + \mathcal{N}_{1,0}(\epsilon U_a) = \epsilon^4 (r_2^1, r_2^2)^T.$$

Assuming that the estimates

$$(4.9) \quad 1 - \epsilon \sup_{t \in [0, T_0/\epsilon]} |\zeta_a(t)|_\infty > 0 \quad \forall \epsilon \leq \epsilon_0 \text{ with some } \epsilon_0 \leq 1$$

and

$$(4.10) \quad \sup_{t \in [0, T_0/\epsilon]} \mathcal{E}_1^N(\epsilon U_a(t)) \leq c_a, \quad |(r_2^1, \mathfrak{P}r_2^2)|_{L^\infty([0, T_0/\epsilon]; H^N)} \leq \epsilon^{-d/2} c_a$$

are satisfied (with $T_0, c_a > 0$ depending only on U_a), we obtain from the stability part of the theorem, in its version for $\epsilon = 1$, the estimate for the error $\mathfrak{e} = \epsilon U - \epsilon U_a$

$$(4.11) \quad \mathcal{E}_1^{N-1}(\mathfrak{e}(t))^{1/2} \leq C(c_2, c_a) \left(\mathcal{E}_1^{N-1}(\mathfrak{e}(0))^{1/2} + t \epsilon^{4-d/2} c_a \right)$$

for $\epsilon \leq \epsilon_0$ and $t \leq T_*/\epsilon$ with $T_* = \min\{T, T_0\}$ and the T, c_2 of Theorem 4.1.

We derive now sufficient conditions on the approximation U_a given by (3.14) and on the residuals r_2^1, r_2^2 given by (3.16), such that the estimates (4.9), (4.10) are satisfied. From the form of U_a and of the residuals it is clear that actually we need conditions on the macroscopic functions comprising them. As these functions are determined through classical partial differential equations, we prefer the conditions to be expressed rather in terms of $|\cdot|_{H^s}$ -norms than in terms of the energy norm \mathcal{E}_1^N or $|\mathfrak{P} \cdot|_{H^s}$.

Denoting with $\tilde{\zeta}_a, \tilde{\psi}_a$ the vectors of all macroscopic functions of ζ_a, ψ_a , respectively, we obtain from (4.7) and (2.8)

$$(4.12) \quad \begin{aligned} \mathcal{E}_1^N(\epsilon U_a) &\leq C \left(h_{\min}^{-1}, \mu_{\max}, |\epsilon \zeta_a|_{H^{t_0+1}}, |\epsilon \nabla \psi_a|_{H^{t_0}} \right) (|\epsilon \zeta_a|_{H^{N+1}}^2 + |\epsilon \nabla \psi_a|_{H^N}^2) \\ &\leq C \left(h_{\min}^{-1}, \mu_{\max}, |\tilde{\zeta}_a|_{H^{t_0+1}}, |\tilde{\psi}_a|_{H^{t_0+1}}, |\xi_j| \right) (|\tilde{\zeta}_a|_{H^{N+1}}^2 + |\tilde{\psi}_a|_{H^{N+1}}^2) \end{aligned}$$

for $\epsilon \leq 1$, and, analogously, for (3.16), recalling (2.9), (2.10), and with (4.6),

$$(4.13) \quad \begin{aligned} \epsilon^{d/2} |(r_2^1, \mathfrak{P}r_2^2)|_{H^N} &\leq C(h_{\min}^{-1}, \mu_{\max}, |\xi_j|, |\tilde{\zeta}_a|_{H^{N+3}}, |\tilde{\psi}_0|_{H^{N+4}}, |\tilde{\psi}_1|_{H^{N+3}}, |\tilde{\psi}_2|_{H^{N+2}}, \\ &\quad |\partial'_t \tilde{\zeta}_{2j}|_{H^N}, |\partial'_t \tilde{\zeta}_{2ji}|_{H^N}, |\partial'_t \tilde{\zeta}_{2jik}|_{H^N}, |\partial'_t \tilde{\zeta}_{20}|_{H^N}, |\partial'_t \tilde{\psi}_{2ji}|_{H^{N+1}}, |\partial'_t \tilde{\psi}_{2jik}|_{H^{N+1}}). \end{aligned}$$

By a careful count of derivatives in the formulas for the macroscopic functions of U_a as obtained in Section 3, we obtain from standard results of qualitative theory (see, e.g., [19, §2.1, §7.2]) for the linear homogeneous and inhomogeneous transport equations and the linear inhomogeneous wave equation of the system (3.17) that, for initial data of the form

$$\psi_{0j}^0 \in H^{s+4}(\mathbb{R}^d), \quad (\psi_{00}^0, \partial'_t \psi_{00}^0) \in H^{s+3} \times H^{s+2}(\mathbb{R}^d), \quad \psi_{1j}^0 \in H^{s+2}(\mathbb{R}^d)$$

with $s \in \mathbb{N}$, $s \geq 2 > t_0 = 3/2$, we have for every $T_0 > 0$ and $t' \leq T_0$ the estimates

$$|(\tilde{\zeta}_0, \tilde{\psi}_0)|_{H^s} \leq C(T_0, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^s}, |(\psi_{00}^0, \partial'_t \psi_{00}^0)|_{H^s \times H^{s-1}}),$$

$$|(\tilde{\zeta}_1, \tilde{\psi}_1)|_{H^s} \leq C(T_0, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^{s+2}}, |(\psi_{00}^0, \partial_t' \psi_{00}^0)|_{H^{s+1} \times H^s}, |\psi_{1j}^0|_{H^s}),$$

$$\begin{aligned} & |(\tilde{\zeta}_2, \tilde{\psi}_2)|_{H^s}, |\partial_t' \tilde{\zeta}_{2ji}|_{H^s}, |\partial_t' \tilde{\zeta}_{2jik}|_{H^s}, |\partial_t' \tilde{\zeta}_{20}|_{H^s}, |\partial_t' \tilde{\psi}_{2ji}|_{H^s}, |\partial_t' \tilde{\psi}_{2jik}|_{H^s} \\ & \leq C(T_0, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^{s+3}}, |(\psi_{00}^0, \partial_t' \psi_{00}^0)|_{H^{s+2} \times H^{s+1}}, |\psi_{1j}^0|_{H^{s+1}}), \end{aligned}$$

$$|\partial_t' \tilde{\zeta}_{2j}|_{H^s} \leq C(T_0, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^{s+4}}, |(\psi_{00}^0, \partial_t' \psi_{00}^0)|_{H^{s+3} \times H^{s+2}}, |\psi_{1j}^0|_{H^{s+2}}).$$

Hence, we obtain from (4.12) and (4.13) that, for initial data

$$(4.14) \quad \psi_{0j}^0 \in H^{N+6}(\mathbb{R}^d), \quad (\psi_{00}^0, \partial_t' \psi_{00}^0) \in H^{N+5} \times H^{N+4}(\mathbb{R}^d), \quad \psi_{1j}^0 \in H^{N+4}(\mathbb{R}^d)$$

with $N \in \mathbb{N}$ as in Theorem 4.1, the approximation U_a of (3.14) satisfies for every $T_0 > 0$ the assumptions (4.10) with

$$c_a = C(T_0, h_{\min}^{-1}, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^{N+6}}, |(\psi_{00}^0, \partial_t' \psi_{00}^0)|_{H^{N+5} \times H^{N+4}}, |\psi_{1j}^0|_{H^{N+4}}),$$

where, since $|\zeta_a(t)|_\infty \leq C|\tilde{\zeta}_a(t')|_\infty \leq C|\tilde{\zeta}_a(t')|_{H^{t_0}}$, we can find for every $T_0 > 0$ some $0 < \epsilon_0 \leq 1$ such that (4.9) is satisfied uniformly for $\epsilon \leq \epsilon_0$, with $h_{\min} > 0$ as a lower bound. Thus, for the initial data (4.14) and sufficiently small ϵ_0 , the error estimate (4.11) holds indeed true.

If one wants to reformulate the estimate (4.11) in terms of Sobolev norms, one needs to estimate its right-hand side from above and its left-hand side from below by such norms. To this end, since

$$(4.15) \quad \mathbf{e} = \epsilon U - \epsilon U_a = \epsilon U - \epsilon U_{a,1} - \epsilon^3(\zeta_2, \psi_2)^T,$$

we obtain from (4.7) for $\epsilon = 1$ and $U^0 = (\zeta^0, \psi^0)^T \in H^{N+1} \times \dot{H}^{N+1}(\mathbb{R}^d)$

$$\mathcal{E}_1^{N-1}(\mathbf{e}(0))^{1/2} \leq C(c_2, c_a) \left(|\epsilon U^0 - \epsilon U_{a,1}(0, \cdot)|_{H^N \times \dot{H}^N} + \epsilon^{3-d/2} c_a \right)$$

with c_2 as in Theorem 4.1 and c_a as above. Moreover, since we get from (4.2)

$$(4.16) \quad |\nabla u|_{H^s} \leq (1 + \sqrt{\mu})^{1/2} |\mathfrak{P}u|_{H^{s+1/2}}, \quad s \in \mathbb{R}, \quad \mu \geq 0,$$

and

$$(4.17) \quad |\mathfrak{P}u|_{H^s} \leq \max\{1, \mu^{-1/4}\} |\nabla u|_{H^{s-1/2}} \leq |u|_{H^{s+1/2}}, \quad s \in \mathbb{R}, \quad \mu \geq 1,$$

we obtain from (4.6), (4.1) and Lemma 2.3(1)

$$\begin{aligned} & |\nabla \psi|_{H^{N-1}} \leq C(\mu_{\max}) |\mathfrak{P}\psi|_{H^{N-1/2}} \\ & \leq C(\mu_{\max}) \sum_{|\alpha| \leq N-1} \left(|\mathfrak{P}\psi_{\epsilon,(\alpha)}|_{H^1} + |\mathfrak{P}((w[\epsilon\zeta]\epsilon\psi)\partial^\alpha \zeta)|_{H^{1/2}} \right) \\ & \leq C(\mu_{\max}) \left(\sum_{|\alpha| \leq N} |\mathfrak{P}\psi_{\epsilon,(\alpha)}|_2 + \sum_{|\alpha| \leq N-1} \sum_{|\beta|=1} |\mathfrak{P}((\partial^\beta(w[\epsilon\zeta]\epsilon\psi))\partial^\alpha \zeta)|_2 \right. \\ & \quad \left. + \sum_{|\alpha| \leq N-1} |(w[\epsilon\zeta]\epsilon\psi)\partial^\alpha \zeta|_{H^1} \right), \\ & \leq C(\mu_{\max}) \left(\mathcal{E}_\epsilon^N(\zeta, \psi)^{1/2} + \sum_{|\alpha| \leq N-1} \sum_{|\beta|=1} |(\partial^\beta(w[\epsilon\zeta]\epsilon\psi))\partial^\alpha \zeta|_{H^1} \right. \\ & \quad \left. + \sum_{|\alpha| \leq N-1} |w[\epsilon\zeta]\epsilon\psi|_{H^{t_0}} |\partial^\alpha \zeta|_{H^1} \right), \\ & \leq C(\mu_{\max}) |w[\epsilon\zeta](\epsilon\psi)|_{H^{t_0+1}} \mathcal{E}_\epsilon^N(\zeta, \psi)^{1/2}, \end{aligned}$$

and, hence, from Lemma 2.3 (4) and (3), and (2.7), (4.16)

$$|\nabla \psi|_{H^{N-1}} \leq C(h_{\min}^{-1}, \mu_{\max}, |\epsilon \zeta|_{H^{t_0+2}}, |\mathfrak{P}(\epsilon \psi)|_{H^{t_0+3/2}}) \mathcal{E}_\epsilon^N(\zeta, \psi)^{1/2}.$$

Since, obviously, also $|\zeta|_{H^N} \leq \mathcal{E}_\epsilon^N(\zeta, \psi)^{1/2}$, we obtain for $U = (\zeta, \psi)^T$

$$(4.18) \quad |U|_{H^N \times \dot{H}^N} \leq C(h_{\min}^{-1}, \mu_{\max}, |\epsilon \zeta|_{H^{t_0+2}}, |\mathfrak{P}(\epsilon \psi)|_{H^{t_0+3/2}}) \mathcal{E}_\epsilon^N(U)^{1/2},$$

and, in particular, for $\epsilon = 1$ and the error \mathfrak{e} of (4.15)

$$|\mathfrak{e}(t)|_{H^{N-1} \times \dot{H}^{N-1}} \leq C(c_2, c_a) \mathcal{E}_1^{N-1}(\mathfrak{e}(t))^{1/2}$$

with N, c_2 as in Theorem 4.1, and, hence, by the triangle inequality,

$$|\epsilon U(t) - \epsilon U_{a,1}(t, \cdot)|_{H^{N-1} \times \dot{H}^{N-1}} \leq C(c_2, c_a) \mathcal{E}_1^{N-1}(\mathfrak{e}(t))^{1/2} + \epsilon^{3-d/2} c_a.$$

Summarizing the above analysis, we obtain as the main result of this article the following justification theorem.

Theorem 4.2. *Under Notation 1.1 (with $\sigma = \frac{1}{\text{Bo}} = 0$) and its assumptions, let $N \geq t_0 + t_0 \vee 2 + 3/2$ with $t_0 = 3/2 > d/2$, and let*

$$\psi_{0j}^0 \in H^{N+6}(\mathbb{R}^d), \quad (\psi_{00}^0, \partial_t' \psi_{00}^0) \in H^{N+5} \times H^{N+4}(\mathbb{R}^d), \quad \psi_{1j}^0 \in H^{N+4}(\mathbb{R}^d),$$

$j = 1, 2, 3$, be the initial data for the system

$$\begin{cases} \partial_t' \psi_{0j} + \nabla \omega_j \cdot \nabla' \psi_{0j} = 0, \\ \partial_t'^2 \psi_{00} - \sqrt{\mu} \Delta' \psi_{00} = \sum_{j=1}^3 ((\omega_j^4 - |\xi_j|^2) \partial_t' + 2\omega_j \xi_j \cdot \nabla') |\psi_{0j}|^2, \\ \partial_t' \psi_{1j} + \nabla \omega_j \cdot \nabla' \psi_{1j} = E_j \end{cases}$$

with $1 \leq \mu \leq \mu_{\max} < \infty$ and

$$E_j = i \frac{1}{2} \nabla' \cdot \mathcal{H}_\omega(\xi_j) \nabla' \psi_{0j} - i \psi_{0j} \left(\frac{1}{2\omega_j} (\omega_j^4 - |\xi_j|^2) \partial_t' + \xi_j \cdot \nabla' \right) \psi_{00} + \tilde{E}_j,$$

where \tilde{E}_j consists of cubic products of ψ_{0j} , see (3.10). Let also

$$\begin{aligned} U_{a,1}(t, X) &= \sum_{j=1}^3 \left(\frac{i\omega_j(\psi_{0j} + \epsilon \psi_{1j}) + \epsilon \nabla \omega_j \cdot \nabla' \psi_{0j}}{\psi_{0j} + \epsilon \psi_{1j}} \right) (t', X') e^{i(\xi_j \cdot X - \omega_j t)} \\ &\quad + \epsilon \sum_{(j,i) \in I} \begin{pmatrix} \zeta_{1ji} \\ \psi_{1ji} \end{pmatrix} (t', X') e^{i((\xi_j + \xi_i) \cdot X - (\omega_j + \omega_i)t)} + \text{c.c.} + \begin{pmatrix} \epsilon \zeta_{10} \\ \psi_{00} \end{pmatrix} (t', X') \end{aligned}$$

with $0 \leq t' = \epsilon t \leq T_0$, $X' = \epsilon X \in \mathbb{R}^d$, $0 < \epsilon \leq 1$, where ζ_{1ji}, ψ_{1ji} consist of quadratic products of ψ_{0j} , see (3.5), and

$$\zeta_{10} = -\partial_t' \psi_{00} + \sum_j (\omega_j^4 - |\xi_j|^2) |\psi_{0j}|^2.$$

Then, for any $c_0 > 0$ there exists an $\epsilon_0 \in (0, 1]$ and a $T > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and all $U^0 = (\zeta^0, \psi^0)^T \in H^{N+1} \times \dot{H}^{N+1}(\mathbb{R}^d)$ with

$$1 - \epsilon |\zeta^0|_\infty \geq h_{\min} > 0, \quad \mathfrak{a}_\epsilon(U^0) \geq a_0 > 0, \quad |\epsilon U^0 - \epsilon U_{a,1}(0, \cdot)|_{H^N \times \dot{H}^N} \leq c_0 \epsilon^{3-d/2}$$

there exists a unique solution $U = (\zeta, \psi)^T \in E_{1,T/\epsilon}^N$ to the water-waves problem

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta] \psi = 0, \\ \partial_t \psi + \zeta + \frac{1}{2} |\nabla \psi|^2 - \frac{(\mathcal{G}[\zeta] \psi + \nabla \zeta \cdot \nabla \psi)^2}{2(1 + |\nabla \zeta|^2)} = 0, \end{cases} \quad U(0) = \epsilon U^0,$$

which satisfies for all $t \leq T_*/\epsilon$ with $T_* = \min\{T, T_0\}$ the estimate

$$|U(t) - \epsilon U_{a,1}(t, \cdot)|_{H^{N-1} \times \dot{H}^{N-1}} \leq C(c, c_a, c_0) \epsilon^{3-d/2},$$

where

$$\begin{aligned} c &= C(|U^0|_{H^{N+1} \times \dot{H}^{N+1}}, h_{\min}^{-1}, a_0^{-1}, \mu_{\max}), \\ c_a &= C(T_0, h_{\min}^{-1}, \mu_{\max}, |\xi_j|, |\psi_{0j}^0|_{H^{N+6}}, |(\psi_{00}^0, \partial_t' \psi_{00}^0)|_{H^{N+5} \times H^{N+4}}, |\psi_{1j}^0|_{H^{N+4}}). \end{aligned}$$

We conclude this article, with some comments on our justification result.

REMARK 4.4.

- (1) For $d = 1, 2$, the theorem holds true with $N = 5$.
- (2) An analogous justification result can be obtained for the transport equations (3.2) by the leading-order-approximation $\epsilon U_{a,0}$, see Remark 3.2., with an error of order $O(\epsilon^{2-d/2})$.
- (3) The justification relies on the stability of the (original) water-waves equation, see Theorem 4.1, and not on the stability of the derived system. In the case of finite depth, such a stability result for the water-waves equation does not exist for time-scales of higher order, e.g., $O(1/\epsilon^2)$.

Moreover, the well-posedness of the macroscopic linear transport equations and the macroscopic wave equation up to any time T_0 imply that the only restriction on the time of validity of the justification result is due to the existence time T for the water-waves problem. However, T_0 influences the constant of the error estimate.

- (4) There is a difference of one order between the Sobolev space in which the initial data of the approximation are assumed to exist and the order of the norm in which the initial distance to the original solution is measured, which is one order higher than the norm of the error for $t > 0$. The latter difference results from the different estimates of the energy norm $\mathcal{E}_\epsilon^N(U)$ from above and from below, see (4.7) and (4.18), while the former one results from the stability result itself, see Theorem 4.1, and for a more detailed analysis, [37]. However, an optimization of the regularity assumptions was not our main focus in this article. Recall, here, that $|U|_{H^N \times \dot{H}^N}^2 = |\zeta|_{H^N}^2 + |\nabla \psi|_{H^{N-1}}^2$.

5. APPENDIX

We give here the definitions of the functions C, D appearing in Proposition 2.4 with the abbreviations of Notation 1.1 and Proposition 2.1(2).

$$\begin{aligned} D_{00} &= \frac{1}{B_0} \Delta' \zeta_{00} - (G_0(\zeta_{00} G_0 \psi_0) + \zeta_{00} \psi_0'') G_0 \psi_0, \\ C_{00} &= -G_1(\zeta_{00} G_0 \psi_0) - G_0(\zeta_{00} G_1 \psi_0) - G_0(\zeta_{00} G_0 \psi_1) \\ &\quad + G_0(\zeta_{00} G_0((\zeta_0 - \zeta_{00}) G_0 \psi_0)) + G_0(\zeta_0 G_0(\zeta_{00} G_0 \psi_0)) \\ &\quad - \nabla' \zeta_{00} \cdot \psi_0' - 2\zeta_{00} \nabla' \cdot \psi_0' - \zeta_{00} \psi_1'' \\ &\quad + \frac{1}{2}(\zeta_{00}(2\zeta_0 - \zeta_{00}) G_0 \psi_0)'' + \frac{1}{2} G_0(\zeta_{00}(2\zeta_0 - \zeta_{00}) \psi_0''), \end{aligned}$$

$$\begin{aligned}
D_0 &= \sum_j (\overline{i\psi_{0j}}(\xi_j - g_j g'_j) \cdot \nabla' \psi_{0j} + (-|\xi_j|^2 + g_j^2) \overline{\psi_{0j}} \psi_{1j}) + \text{c.c.} + D_{00}, \\
C_0 &= \sum_j i\xi_j \cdot \nabla' (\zeta_{0j} \overline{\psi_{0j}}) + \text{c.c.} + C_{00}, \\
D_{jj} &= -i\psi_{0j}(\xi_j + g_j g'_j) \cdot \nabla' \psi_{0j} + (|\xi_j|^2 + g_j^2) \psi_{0j} \psi_{1j}, \\
C_{jj} &= i(g_j g'_{jj} - \xi_j) \cdot \nabla' (\zeta_{0j} \psi_{0j}) + i\zeta_{0j}(g_{jj} g'_j - 2\xi_j) \cdot \nabla' \psi_{0j} \\
&\quad - (g_j g_{jj} - 2|\xi_j|^2)(\psi_{0j} \zeta_{1j} + \zeta_{0j} \psi_{1j}) \quad \text{for } j \in J, \\
D_{ji} &= -i\psi_{0i}(\xi_i + g_i g'_i) \cdot \nabla' \psi_{0j} - i\psi_{0j}(\xi_j + g_j g'_j) \cdot \nabla' \psi_{0i} \\
&\quad + (\xi_j \cdot \xi_i + g_j g_i)(\psi_{0i} \psi_{1j} + \psi_{0j} \psi_{1i}), \\
C_{ji} &= i(g_i g'_{ji} - \xi_i) \cdot \nabla' (\zeta_{0j} \psi_{0i}) + i(g_j g'_{ji} - \xi_j) \cdot \nabla' (\zeta_{0i} \psi_{0j}) \\
&\quad + i\zeta_{0j}(g_{ji} g'_i - \xi_{ji}) \cdot \nabla' \psi_{0i} + i\zeta_{0i}(g_{ji} g'_j - \xi_{ji}) \cdot \nabla' \psi_{0j} \\
&\quad - (g_i g_{ji} - \xi_i \cdot \xi_{ji})(\psi_{0i} \zeta_{1j} + \zeta_{0j} \psi_{1i}) \\
&\quad - (g_j g_{ji} - \xi_j \cdot \xi_{ji})(\psi_{0j} \zeta_{1i} + \zeta_{0i} \psi_{1j}) \quad \text{for } (j, i) \in I_<, \\
D_{j,-i} &= i\overline{\psi_{0i}}(\xi_i - g_i g'_i) \cdot \nabla' \psi_{0j} - i\psi_{0j}(\xi_j - g_j g'_j) \cdot \nabla' \overline{\psi_{0i}} \\
&\quad + (-\xi_j \cdot \xi_i + g_j g_i)(\overline{\psi_{0i}} \psi_{1j} + \psi_{0j} \overline{\psi_{1i}}), \\
C_{j,-i} &= i(g_i g'_{j,-i} + \xi_i) \cdot \nabla' (\zeta_{0j} \overline{\psi_{0i}}) + i(g_j g'_{j,-i} - \xi_j) \cdot \nabla' (\overline{\zeta_{0i}} \psi_{0j}) \\
&\quad - i\zeta_{0j}(g_{j,-i} g'_i + \xi_{j,-i}) \cdot \nabla' \overline{\psi_{0i}} + i\overline{\zeta_{0i}}(g_{j,-i} g'_j - \xi_{j,-i}) \cdot \nabla' \psi_{0j} \\
&\quad - (g_i g_{j,-i} + \xi_i \cdot \xi_{j,-i})(\overline{\psi_{0i}} \zeta_{1j} + \zeta_{0j} \overline{\psi_{1i}}) \\
&\quad - (g_j g_{j,-i} - \xi_j \cdot \xi_{j,-i})(\psi_{0j} \overline{\zeta_{1i}} + \overline{\zeta_{0i}} \psi_{1j}) \quad \text{for } (j, i) \in I_<, \\
D_j &= -d_j^{(1)} + d_j^{(2)} + g_j (\sum_i |\xi_i|^2 \zeta_{0i} \overline{\psi_{0i}} + \text{c.c.}) \psi_{0j} \\
&\quad + \frac{1}{\text{Bo}} \left(i\xi_j \cdot d_j^{(3)} + |\xi_j|^2 (\sum_i |\xi_i|^2 |\zeta_{0i}|^2) \zeta_{0j} \right), \\
C_j &= -g_j (d_j^{(4)} + d_j^{(5)}) - (d_j^{(6)} + d_j^{(7)} + d_j^{(8)} + d_j^{(9)}) \\
&\quad - \frac{1}{2} |\xi_j|^2 d_j^{(10)} + \frac{1}{2} g_j d_j^{(11)} - 2|\xi_j|^2 g_j (\sum_i |\zeta_{0i}|^2) \psi_{0j} \quad \text{for } j \in J, \\
D_{jik} &= -d_{jik}^{(1)} + d_{jik}^{(2)} + \frac{1}{\text{Bo}} i\xi_{jik} \cdot d_{jik}^{(3)}, \\
C_{jik} &= -g_{jik} (d_{jik}^{(4)} + d_{jik}^{(5)}) - (d_{jik}^{(6)} + d_{jik}^{(7)} + d_{jik}^{(8)} + d_{jik}^{(9)}) \\
&\quad - \frac{1}{2} |\xi_{jik}|^2 d_{jik}^{(10)} + \frac{1}{2} g_{jik} d_{jik}^{(11)} \quad \text{for } (j, i, k) \in K.
\end{aligned}$$

The $d_j^{(n)}$, $d_{jik}^{(n)}$ ($n \in \{1, \dots, 11\}$) can be calculated from $c_{ji}^{(n)}$, $a_k^{(n)}$ by the formulas

$$\begin{aligned}
d_1 &= \sum_i c_{1i} \bar{a}_i + \sum_{i=2,3} c_{1,-i} a_i, \\
d_2 &= c_{12} \bar{a}_1 + c_{22} \bar{a}_2 + c_{23} \bar{a}_3 + \bar{c}_{1,-2} a_1 + c_{2,-3} a_3, \\
d_3 &= \sum_j c_{j3} \bar{a}_j + \sum_{j=1,2} \bar{c}_{j,-3} a_j, \\
d_{jjj} &= c_{jj} a_j \quad \text{for } j \in J, \\
d_{jji} &= c_{jj} a_i + c_{ji} a_j, \quad d_{iij} = c_{ii} a_j + c_{ji} a_i \quad \text{for } (j, i) \in I_<, \\
d_{jj,-i} &= c_{jj} \bar{a}_i + c_{j,-i} a_j, \quad d_{i,-j} = c_{ii} \bar{a}_j + \bar{c}_{j,-i} a_i \quad \text{for } (j, i) \in I_<, \\
d_{123} &= c_{23} a_1 + c_{13} a_2 + c_{12} a_3, \\
d_{12,-3} &= c_{2,-3} a_1 + c_{1,-3} a_2 + c_{12} \bar{a}_3,
\end{aligned}$$

$$\begin{aligned} d_{13,-2} &= \bar{c}_{2,-3}a_1 + c_{13}\bar{a}_2 + c_{1,-2}a_3, \\ d_{23,-1} &= c_{23}\bar{a}_1 + \bar{c}_{1,-3}a_2 + \bar{c}_{1,-2}a_3 \end{aligned}$$

with

$$\begin{aligned} c_{ji}^{(1)} &= i\xi_{ji}\psi_{1ji}, & a_k^{(1)} &= i\xi_k\psi_{0k}, \\ c_{ji}^{(2)} &= \gamma_{ji}^{(1)}, & a_k^{(2)} &= g_k\psi_{0k}, \\ c_{ji}^{(3)} &= \gamma_{ji}^{(2)}, & a_k^{(3)} &= i\xi_k\zeta_{0k}, \\ c_{ji}^{(4)} &= \zeta_{1ji}, & a_k^{(4)} &= g_k\psi_{0k}, \\ c_{ji}^{(5)} &= g_{ji}(\psi_{1ji} - \gamma_{ji}^{(3)}), & a_k^{(5)} &= \zeta_{0k}, \\ c_{ji}^{(6)} &= i\xi_{ji}\zeta_{1ji}, & a_k^{(6)} &= i\xi_k\psi_{0k}, \\ c_{ji}^{(7)} &= \zeta_{1ji}, & a_k^{(7)} &= -|\xi_k|^2\psi_{0k}, \\ c_{ji}^{(8)} &= i\xi_{ji}\psi_{1ji}, & a_k^{(8)} &= i\xi_k\zeta_{0k}, \\ c_{ji}^{(9)} &= -|\xi_{ji}|^2\psi_{1ji}, & a_k^{(9)} &= \zeta_{0k}, \\ c_{ji}^{(10)} &= \gamma_{ji}^{(4)}, & a_k^{(10)} &= g_k\psi_{0k}, \\ c_{ji}^{(11)} &= \gamma_{ji}^{(4)}, & a_k^{(11)} &= -|\xi_k|^2\psi_{0k} \end{aligned}$$

for $(j, i) \in I$, $k \in J$, where

$$\begin{aligned} \gamma_{jj}^{(1)} &= g_{jj}\psi_{1jj} + (|\xi_j|^2 - g_{jj}g_j)\zeta_{0j}\psi_{0j}, \\ \gamma_{jj}^{(2)} &= \frac{1}{2}|\xi_j|^2\zeta_{0j}^2, \quad \gamma_{jj}^{(3)} = g_j\zeta_{0j}\psi_{0j}, \quad \gamma_{jj}^{(4)} = \zeta_{0j}^2 \quad \text{for } j \in J, \\ \gamma_{ji}^{(1)} &= g_{ji}\psi_{1ji} + (|\xi_i|^2 - g_{ji}g_i)\zeta_{0j}\psi_{0i} + (|\xi_j|^2 - g_{ji}g_j)\zeta_{0i}\psi_{0j}, \\ \gamma_{ji}^{(2)} &= \xi_j \cdot \xi_i\zeta_{0j}\zeta_{0i}, \quad \gamma_{ji}^{(3)} = g_i\zeta_{0j}\psi_{0i} + g_j\zeta_{0i}\psi_{0j}, \quad \gamma_{ji}^{(4)} = 2\zeta_{0j}\zeta_{0i}, \quad (j, i) \in I_<, \\ \gamma_{j,-i}^{(1)} &= g_{j,-i}\psi_{1j,-i} + (|\xi_i|^2 - g_{j,-i}g_i)\zeta_{0j}\overline{\psi_{0i}} + (|\xi_j|^2 - g_{j,-i}g_j)\overline{\zeta_{0i}}\psi_{0j}, \\ \gamma_{j,-i}^{(2)} &= -\xi_j \cdot \xi_i\zeta_{0j}\overline{\zeta_{0i}}, \quad \gamma_{j,-i}^{(3)} = g_i\zeta_{0j}\overline{\psi_{0i}} + g_j\overline{\zeta_{0i}}\psi_{0j}, \quad \gamma_{j,-i}^{(4)} = 2\zeta_{0j}\overline{\zeta_{0i}}, \quad (j, i) \in I_<. \end{aligned}$$

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